Inverse problems on Riemann surfaces

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CHAPTER 1

Introduction

In this course we will provide a complete solution for the inverse boundary value problem for the elliptic operator $\Delta + V$ on a Riemann surface. That is, on a Riemann surface M we will recover the coefficient V from boundary data of the solutions of the equation $(\Delta + V)u = 0$.

This problem turns out to have rich topological and geometrical structure. In this course we will see where these geometrical considerations arise and what are the tools needed to understand these problems. The course also gives an introduction to Riemann surfaces (no prior knowledge is assumed).

The prerequisites for reading these lecture notes include a basic knowledge of real and complex analysis and also functional analysis. Chapters 1–11 and 14 of Rudin [**10**] should be more than sufficient for this purpose.

Familiarity with differential geometry, Riemannian geometry, and partial differential equations will also be helpful, although we will review (mostly without proofs) all the results that are needed. Our basic reference for differential geometry and the theory of smooth manifolds is Lee [**7**]. A good introduction to Riemannian geometry is given in another book of Lee [**8**]. However, a more useful reference for our purposes is Taylor [**12**] which contains many results in Riemannian geometry from an analysis point of view. We do not assume knowledge of the theory of Riemann surfaces, and the required results are either proved or stated without proof. For background material on partial differential equations, we refer to Evans [**1**] and Taylor [**12**].

We do not assume familiarity with inverse problems, and all relevant results will be proved in full. For more information on the Calderón problem, there is the short introduction Salo [**11**] and the partially completed textbook Feldman-Uhlmann [**2**]. For the main part of the course, which gives a solution to Calderón's inverse problem on Riemann surfaces, we will follow the two articles of Guillarmou and Tzou [**3**], [**4**].

CHAPTER 2

The Calderón problem on Riemann surfaces

2.1. Basic setup

Let (M, g) be a smooth 2D Riemannian manifold with boundary ∂M. We denote by Δ_g the Laplace-Beltrami operator on M. If V is a smooth function on M, consider the Schrödinger equation

$$(-\Delta_g + V)u = 0 \qquad \text{in } M.$$

We consider the inverse problem of determining the function V from the knowledge of boundary measurements of solutions to the Schrödinger equation (the manifold (M, g) is assumed to be known).

This is an analog of Calderón's inverse problem of determining an electrical conductivity γ from boundary measurements for the conductivity equation $\nabla \cdot \gamma \nabla u = 0$ in a bounded domain in $\mathbb{R}^2$. In fact, our result will give a solution of Calderón's problem in 2D (for relatively smooth γ) as a corollary by a standard reduction of the conductivity equation to a Schrödinger equation.

More precisely the boundary measurements are given by the *Cauchy data set* for H^2 solutions of the Schrödinger equation, defined by

$$C_V := \{(u|_{\partial M}, \partial_\nu u|_{\partial M})\,;\, u \in H^2(M) \text{ satisfies } (-\Delta_g + V)u = 0 \text{ in } M\}.$$

If 0 is not a Dirichlet eigenvalue of $-\Delta_g + V$ in M, one also has the Dirichlet-to-Neumann map

$$\Lambda_V : H^{3/2}(\partial M) \to H^{1/2}(\partial M), \quad f \mapsto \partial_\nu u_f|_{\partial M}$$

where u_f is the unique H^2 solution of the Schrödinger equation with $u_f|_{\partial M} = f$. In this case C_V is just the graph of Λ_V, and knowing C_V is equivalent to knowing Λ_V.

We will actually consider boundary measurements taken on an open subset Γ of ∂M. The *Cauchy data set on* Γ is

$$C_V\Gamma := \{(u|_\Gamma, \partial_\nu u|_\Gamma)\,;\, u \in H^2(M) \text{ satisfies } (-\Delta_g + V)u = 0 \text{ in } M \text{ and } u|_{\partial M \setminus \Gamma} = 0\}.$$

Again, if 0 is not a Dirichlet eigenvalue then knowing C_V^Γ is equivalent to knowing $\Lambda_V f|_\Gamma$ for all f in $H^2(\partial M)$ such that $\mathrm{supp}(f) \subseteq \overline{\Gamma}$.

The main result proved during this course is the following.

THEOREM 2.1.1. *(Guillarmou-Tzou 2009) Let (M,g) be a compact oriented 2D manifold with smooth boundary, and let V_1 and V_2 be smooth functions in M. If Γ is any open subset of ∂M, then $C_{V_1} = C_{V_2}$ implies that $V_1 = V_2$.*

As mentioned, this implies the corresponding uniqueness result with partial data for the conductivity equation in 2D. Recall that the Dirichlet-to-Neumann map Λ_γ for the conductivity equation maps f to $\gamma \partial_\nu u|_{\partial\Omega}$ where u solves $\nabla \cdot (\gamma \nabla u) = 0$ in Ω and $u|_{\partial\Omega} = f$.

THEOREM 2.1.2. *(Imanuvilov-Uhlmann-Yamamoto 2009) Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary, and let γ_1 and γ_2 be two positive smooth functions on $\overline{\Omega}$. If Γ is any open subset of ∂M and if*

$$\Lambda_{\gamma_1} f|_\Gamma = \Lambda_{\gamma_2} f|_\Gamma \quad \text{for all } f \in H^{3/2}(\partial\Omega) \text{ with } \mathrm{supp}(f) \subseteq \Gamma,$$

then $\gamma_1 = \gamma_2$.

We now describe some ideas of the proof. The starting point is an integral identity which relates the condition $C_{V_1} = C_{V_2}$ to an integral involving the difference of the potentials.

LEMMA 2.1.3. *Let V_1 and V_2 be two smooth functions on M. If $C_{V_1}^\Gamma = C_{V_2}^\Gamma$, then one has*

$$\int_M (V_1 - V_2) u_1 u_2 \, dV_g = 0$$

for any u_j which are H^2 solutions of $(-\Delta_g + V_j) u_j = 0$ in M and satisfy $u_j|_{\partial M \setminus \Gamma} = 0$, $j = 1, 2$.

Thus, to prove that $V_1 = V_2$, it is enough to produce many solutions u_j to the Schrödinger equation which vanish on $\partial M \setminus \Gamma$. More precisely, we want that the set of products $\{u_1 u_2\}$ for such solutions is dense in, say, $L^1(M)$.

The solutions that will be used are particular *complex geometrical optics* solutions, which appear in most studies of inverse problems of this type. Following an idea of Bukhgeim, the solutions will have the form

$$u = e^{\tau\Phi} m$$

where τ is a large parameter, and m will have explicit form as $\tau \to \infty$. The point is that Φ will be a holomorphic function in M having a nondegenerate critical point at (or near) a given point p of M. The model case in the complex plane is $\Phi(z) = (z - z_0)^2$ which has a critical point at z_0. The proof that V_1 is equal to V_2 will be modelled after the idea that the condition

$$\int_{\mathbb{C}} e^{i\tau \mathrm{Im}((z-z_0)^2)} f \, dA = 0,$$

for a smooth compactly supported function f and for all $\tau > 0$, will imply that $f(z_0) = 0$ by the method of stationary phase.

One challenge is that it is not easy to find explicit holomorphic functions on a given Riemann surface. However, the existence of such functions is guaranteed by general results such as the Riemann-Roch theorem which will be used heavily in the argument. Another complication is that the holomorphic functions that one obtains may have many degenerate critical points. We will need to approximate such functions by holomorphic functions having only nondegenerate critical points, in order to be able to use the stationary phase method (this is an analog of the Morse theory fact that smooth functions can be approximated by Morse functions).

Furthermore, when dealing with partial data, we need to carry out the constructions in such a way that the holomorphic functions will be purely real on the inaccessible part $\partial M \setminus \Gamma$ of the boundary. This will allow to construct solutions vanishing on the inaccessible part. For this, we employ a version of the Riemann-Roch theorem on manifolds with boundary, which implies the existence of holomorphic functions with prescribed zeros and critical points provided that the functions are allowed to have large winding numbers (expressed in terms of the Maslov index of a suitable bundle) on the possibly small subset Γ of ∂M.

2.2. Isothermal coordinates

In this section we show that on any oriented 2D Riemannian manifold, there is a system of coordinate charts such that the metric g becomes a scalar multiple of the identity matrix in these coordinates. These charts are called *isothermal coordinates* or *conformal coordinates* (also *holomorphic coordinates*). The main result is as follows.

THEOREM 2.2.1. *(Isothermal coordinates) If (M, g) is an orientable 2D manifold, there is a system of conformal coordinate charts $(U_\alpha, \varphi_\alpha)$. In particular, for each α*

$$(\varphi_\alpha^{-1})^* g = \lambda_\alpha e$$

where λ_α is a smooth positive function in $\varphi_\alpha(U_\alpha)$, and if $U_\alpha \cap U_\beta \neq \emptyset$ then $\varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic $\varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$.

Here conformality of the charts means that each φ_α is a conformal transformation, in the sense of the next definition, between (U_α, g) and $(\varphi_\alpha(U_\alpha), e)$ where e is the Euclidean metric in $\mathbb{R}^2$.

DEFINITION. Two metrics g_1 and g_2 on a manifold M are called *conformal* if $g_2 = \lambda g_1$ for a smooth positive function λ on M. A diffeomorphism $f : (M, g) \to (M', g')$ is called a *conformal transformation* if $f^* g'$ is conformal to g, that is,

$$f^* g' = \lambda g.$$

Two Riemannian manifolds are called conformal if there is a conformal transformation between them.

We relate this definition of conformality to the standard one in complex analysis by defining the angle $\theta_g(v, w) \in [0, \pi]$ between two tangent vectors $v, w \in T_p M$ by

$$\cos \theta_g(v, w) = \frac{\langle v, w \rangle_g}{|v|_g |w|_g}.$$

LEMMA 2.2.2. *(Conformal = angle-preserving) Let $f : (M, g) \to (M', g')$ be a diffeomorphism. The following are equivalent.*

1. *f is a conformal transformation.*
2. *f preserves angles in the sense that $\theta_g(v, w) = \theta_{g'}(f_* v, f_* w)$.*
3. *f^* maps any orthonormal basis of $T^*_{f(p)} M'$ to an orthogonal basis, whose vectors all have the same length, of $T^*_p M$.*

PROOF. Exercise. □

The two dimensional case is special because of the classical fact that orientation preserving conformal maps are holomorphic. The proof is given for completeness.

LEMMA 2.2.3. *(Conformal = holomorphic) Let Ω and $\tilde{\Omega}$ be open sets in $\mathbb{R}^2$. An orientation preserving map $f : \Omega \to \Omega'$ is conformal iff it is holomorphic and bijective.*

PROOF. We use complex notation and write $z = x+iy$, $f = u+iv$. If f is conformal then it is bijective and $f^*e = \lambda e$. The last condition means that for all $z \in \Omega$ and for $v, w \in \mathbb{R}^2$,

$$\lambda(z)v \cdot w = (f_*v) \cdot (f_*w) = Df(z)v \cdot Df(z)w = Df(z)^t Df(z)v \cdot w.$$

Since $Df(z) = \left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)$, this implies

$$\begin{pmatrix} u_x^2 + v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & u_y^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

Thus the vectors $(u_x \ v_x)^t$ and $(u_y \ v_y)^t$ are orthogonal and have the same length. Since f is orientation preserving so $\det Df > 0$, we must have

$$u_x = v_y, \quad u_y = -v_x.$$

This shows that f is holomorphic. The converse follows by running the argument backwards. $\square$

Note that the Cauchy-Riemann equations for u and v can also be written as

$$dv = *du$$

since $du = u_x\,dx + u_y\,dy$, $dv = v_x\,dx + v_y\,dy$, and since the Hodge star satisfies $*dx = dy$, $*dy = -dx$ (therefore $*$ corresponds to the counterclockwise rotation by 90°).

We are now ready to prove the existence of isothermal coordinates on 2D manifolds.

PROOF OF THEOREM 2.2.1. It is enough to show that for any point $p \in M$, there is a neighborhood U and an orientation preserving diffeomorphism $\varphi : U \to \tilde{U}$ onto an open set in $\mathbb{R}^2$ such that

$$\varphi^*(dx) \text{ and } \varphi^*(dy) \text{ are orthogonal and have the same length.}$$

Then φ is conformal by Lemma 2.2.2 and simple linear algebra, and the condition that the $\varphi_\beta \circ \varphi_\alpha^{-1}$ are holomorphic follows from Lemma 2.2.3 and the fact that compositions and inverses of conformal maps are conformal.

We write $\varphi = (u, v)$ for the first and second coordinates, and note that $\varphi^*(dx) = du$, $\varphi^*(dy) = dv$. Since the Hodge star on 1-forms is rotation by 90°, φ satisfies the required condition provided that

$$dv = *du.$$

These are the Cauchy-Riemann equations in (M, g), and they state that u and v should be *conjugate harmonic functions*: taking $*d$ implies

$$\Delta_g u = *d * du = *d(dv) = 0$$

and taking $*d*$ gives $\Delta_g v = 0$.

Now, suppose u is a harmonic function defined in a simply connected neighborhood U of p. Then there always exists a harmonic conjugate: the 1-form $*du$ is closed since $d(*du) = -*(\Delta_g u) = 0$, and by Poincaré's lemma (which in the case of 1-forms in 2D is an easy result from vector calculus) there is a smooth function v in U with $*du = dv$. If additionally $du(p) \neq 0$ then one also has $dv(p) \neq 0$, and the inverse function theorem implies that $\varphi = (u, v)$ is a diffeomorphism near p. Thus the theorem will follow from the next result below. $\square$

LEMMA 2.2.4. *If (M, g) is a 2D manifold, then for any point $p \in M$ there is a harmonic function u near p satisfying $du(p) \neq 0$.*

PROOF. Let $\varphi = (x^1, x^2)$ be some chart mapping a neighborhood U of p onto the unit disc $\mathbb{D}$ in the plane. For $\varepsilon > 0$, let $\mathbb{D}_\varepsilon$ be the disc of radius ε and let $U_\varepsilon = \varphi^{-1}(\mathbb{D}_\varepsilon)$. We will find a harmonic function u in U_ε approximating the scaled coordinate function x^1/ε, which has nonzero derivative at p.

More precisely, we take u to be the solution of the equation $\Delta_g u = 0$ in U_ε with boundary value $u|_{\partial U_\varepsilon} = x^1/\varepsilon$. In the x coordinates one has $\Delta_g u(x) = |g|^{-1/2}\partial_j(|g|^{1/2} g^{jk} \partial_k u(x))$, and the boundary problem becomes

$$\begin{cases} g^{jk}(x)\partial_{jk}u(x) + b^k(x)\partial_k u(x) = 0 & \text{in } \mathbb{D}_\varepsilon, \\ u|_{\partial \mathbb{D}_\varepsilon} = x^1/\varepsilon. & \end{cases}$$

We define new dilated coordinates $\tilde{x} = x/\varepsilon$ and the corresponding function $\tilde{u}(\tilde{x}) = u(\varepsilon\tilde{x})$. Then $u(x) = \tilde{u}(x/\varepsilon)$, and the above problem is equivalent with

$$\begin{cases} g^{jk}(\varepsilon\tilde{x})\partial_{jk}\tilde{u}(\tilde{x}) + \varepsilon b^k(\varepsilon\tilde{x})\partial_k \tilde{u}(\tilde{x}) = 0 & \text{in } \mathbb{D}, \\ \tilde{u}|_{\partial \mathbb{D}} = \tilde{x}^1. & \end{cases}$$

Thus, $\tilde{u} = \tilde{x}^1 + \varepsilon\tilde{r}$ where $\tilde{r}$ solves

$$\begin{cases} g^{jk}(\varepsilon\tilde{x})\partial_{jk}\tilde{r}(\tilde{x}) + \varepsilon b^k(\varepsilon\tilde{x})\partial_k \tilde{r}(\tilde{x}) = -b^1(\varepsilon\tilde{x}) & \text{in } \mathbb{D}, \\ \tilde{u}|_{\partial \mathbb{D}} = 0. & \end{cases}$$

By elliptic regularity, the last problem has a solution $\tilde{r} \in C^\infty(\overline{\mathbb{D}})$ satisfying

$$\|\tilde{r}\|_{H^3(\mathbb{D})} \leq C\|b^1(\varepsilon\,\cdot\,)\|_{H^1(\mathbb{D})}.$$

The constant C is independent of $\varepsilon \in (0,1]$ since the ellipticity constant of $(g^{jk}(\varepsilon\,\cdot\,))$ and the C^2 norms of $g^{jk}(\varepsilon\,\cdot\,)$ and $b^k(\varepsilon\,\cdot\,)$ are bounded uniformly in ε. Thus by the Sobolev embedding,

$$\|\tilde{r}\|_{C^1(\overline{\mathbb{D}})} \leq C \quad \text{uniformly in } \varepsilon.$$

It follows that $d\tilde{u}(0) \neq 0$ if ε is small enough, which implies that $du(p) \neq 0$ as required. $\square$

2.3. Complex analysis

The existence of isothermal coordinates allows to define a complex structure on any 2D oriented Riemannian manifold (M,g). Many familiar concepts in complex analysis then carry over to this situation. This section will consist of basic definitions and lemmas.

Let $\varphi = (x,y)$ be a conformal chart as in Theorem 2.2.1, and write $z = x+iy$ and $\bar{z} = x - iy$. *The variables $x, y, z, \bar{z}$ will always have this meaning in the rest of Chapter 2.* Writing $\mathbb{C}V = \{v+iw\,;\,v,w \in V\}$ for a real vector space V, we obtain the complexified tangent and cotangent bundles as the disjoint unions

$$\mathbb{C}TM := \bigsqcup_{p\in M} \mathbb{C}T_pM, \qquad \mathbb{C}T^*M := \bigsqcup_{p\in M} \mathbb{C}T_p^*M.$$

Then complex vector fields (resp. 1-forms) are just sections of $\mathbb{C}TM$ (resp. $\mathbb{C}T^*M$). We define the vector fields

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Any complex vector field can locally be written in terms of these two. Extending d to complex differential forms by linearity, we have the corresponding 1-forms

$$dz = dx + i\,dy, \qquad d\bar{z} = dx - i\,dy.$$

The quantities dz and $d\bar{z}$ are the basic 1-forms in terms of which all complex 1-forms can be locally expressed. As an example, the exterior derivative on functions obtains the following form.

LEMMA 2.3.1. *If f is a smooth complex function on M, then*

$$df = \frac{\partial f}{\partial z}\, dz + \frac{\partial f}{\partial \bar{z}}\, d\bar{z}.$$

PROOF. Exercise. $\square$

We extend the Hodge star operator to complex forms by linearity. Since $*dx = dy$ and $*dy = -dx$, we have

$$*dz = -i\, dz, \qquad *d\bar{z} = i\, d\bar{z}.$$

Thus $*$ on complex 1-forms is diagonal in the basis $\{dz, d\bar{z}\}$ and has eigenvalues $\pm i$. This induces a splitting of the complex cotangent bundle

$$\mathbb{C}T^*M = T^*_{(1,0)}M \oplus T^*_{(0,1)}M$$

where we define

$$T^*_{(1,0)}M := \mathrm{Ker}(* + i), \qquad T^*_{(0,1)}M := \mathrm{Ker}(* - i).$$

Locally, $T^*_{(1,0)}M$ consists of multiples of dz and $T^*_{(0,1)}M$ contains multiples of $d\bar{z}$.

We move on to the definition of ∂ and $\overline{\partial}$ operators. First consider the natural projections

$$\pi_{(1,0)} : \mathbb{C}T^*M \to T^*_{(1,0)}M, \qquad \pi_{(0,1)} : \mathbb{C}T^*M \to T^*_{(1,0)}M.$$

DEFINITION. If $f \in C^\infty(M)$ is a complex function, define

$$\partial f := \pi_{(1,0)} df, \qquad \overline{\partial} f := \pi_{(0,1)} df.$$

If ω is a complex 1-form, define

$$\partial \omega := d(\pi_{(0,1)}\omega), \qquad \overline{\partial}\omega := d(\pi_{(1,0)}\omega).$$

LEMMA 2.3.2. *Let f be a smooth complex function and ω a complex 1-form. Locally in the z coordinates, if $\omega = u\, dz + v\, d\bar{z}$ then*

$$\partial f = \frac{\partial f}{\partial z}\, dz, \qquad \overline{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z},$$

$$\partial \omega = \frac{\partial v}{\partial z}\, dz \wedge d\bar{z}, \qquad \overline{\partial}\omega = \frac{\partial u}{\partial \bar{z}}\, d\bar{z} \wedge dz.$$

One has $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$.

PROOF. Exercise. $\square$

We next show how the Laplace-Beltrami operator Δ_g can be factored in terms of the ∂ and $\overline{\partial}$ operators.

LEMMA 2.3.3. *One has for functions*

$$\Delta_g = 2i * \partial\overline{\partial} = -2i * \overline{\partial}\partial.$$

PROOF. We have the decomposition $d = \partial + \overline{\partial}$, which is valid both for functions and 1-forms. Thus

$$\Delta_g = *d * d = *(\partial + \overline{\partial}) * (\partial + \overline{\partial}) = *(\partial * \partial + \partial * \overline{\partial} + \overline{\partial} * \partial + \overline{\partial} * \overline{\partial}).$$

The definition of $T^*_{(1,0)}M$ and $T^*_{(0,1)}M$ implies that for functions

$$*\partial = *\pi_{(1,0)}d = -i\pi_{(1,0)}d = -i\partial,$$
$$*\overline{\partial} = *\pi_{(0,1)}d = i\pi_{(0,1)}d = i\overline{\partial}.$$

Therefore, since $\partial^2 = \overline{\partial}^2 = 0$,

$$\Delta_g = *(\partial * \overline{\partial} + \overline{\partial} * \partial) = i * (\partial\overline{\partial} - \overline{\partial}\partial).$$

The result follows from the fact that $\partial\overline{\partial} + \overline{\partial}\partial = 0$. □

Finally, we define holomorphic functions on a Riemann surface and show several equivalent characterizations.

DEFINITION. If U is an open set in (M, g), we say that $f \in C^\infty(U)$ is holomorphic if $\frac{\partial f}{\partial \bar{z}} = 0$ in U.

LEMMA 2.3.4. *The following are equivalent.*

(a) f *is holomorphic.*
(b) $\overline{\partial} f = 0$.
(c) *For any conformal chart* φ_α, *the function* $f \circ \varphi_\alpha^{-1}$ *is holomorphic in the usual sense.*
(d) *One has* $f = u + iv$ *where* u *and* v *are real valued harmonic functions (so that* $\Delta_g u = \Delta_g v = 0$*) satisfying* $dv = *du$.

PROOF. Exercise. □

An immediate corollary is that the real and imaginary parts of a holomorphic function in (M, g) are harmonic in (M, g).

2.4. Riemann-Roch theorem

In the solution of the inverse problem on Riemann surfaces, we will need to construct special holomorphic functions which have prescribed zeros and poles at certain points. One of the main tools for doing this is a version of the Riemann-Roch theorem for surfaces with boundary. However, for motivation we first discuss this classical result on compact surfaces without boundary. We then consider the Maslov index on real line bundles over ∂M, and state the Riemann-Roch theorem which will actually be used.

Classical Riemann-Roch. Let (M, g) be a compact, connected, oriented 2D Riemannian manifold with no boundary. If f is a meromorphic function on M, we define the *divisor* of f to be the formal sum

$$(f) := \sum_{z_j \text{ zero or pole of } f} \operatorname{ord}(z_j) z_j$$

where

$$\operatorname{ord}(z) := \begin{cases} m, & z \text{ is a zero of order } m, \\ -m, & z \text{ is a pole of order } m. \end{cases}$$

More generally, a *divisor* on M is any function $D : M \to \mathbb{Z}$ such that $D(p) \neq 0$ only for finitely many points p in M. We write formally

$$D = \sum_p D(p)p.$$

The *degree* of a divisor is the integer $\deg(D) := \sum_p D(p)$.

To track the zeros and poles of meromorphic functions at prescribed points, we introduce the vector space

$$L(D) := \{f \text{ meromorphic in } M\,;\ (f) + D \geq 0 \text{ or } f \equiv 0\}.$$

Thus, $f \in L(D)$ means that f has a zero of order $\geq -D(p)$ at p when $D(p) < 0$, and a pole of order $\leq D(p)$ at p when $D(p) > 0$ (and f has no other poles). For example, to have a meromorphic function on M with zeros at points $p_1, \ldots, p_N$, one needs $L(D)$ to be nontrivial for some divisor D with $D(p_1) = \ldots = D(p_N) = -1$.

The Riemann-Roch theorem shows, among other things, that $L(D)$ is always nontrivial if one allows a pole of sufficiently high order at some point. We only state *Riemann's inequality* which follows immediately from the usual formulation of Riemann-Roch (see [**5**, Section 5.4] where also a proof is given).

THEOREM 2.4.1. *(Riemann's inequality) If M has genus k and if D is a divisor on M, then*

$$\dim L(D) \geq \deg(D) - k + 1.$$

In particular, if $\deg(D) \geq k$, then $L(D)$ is nontrivial.

Example. Suppose we want to find a meromorphic function on M with zeros at points $p_1, \ldots, p_N$. If M has genus k, we consider the divisor

$$D = -p_1 - \ldots - p_N + (N+k)p_{N+1}$$

where p_{N+1} is an additional point of M. Then $\deg(D) = k$ so $L(D)$ is nonempty by Riemann's inequality. This shows that we can always find a nontrivial meromorphic function with zeros at $p_1, \ldots, p_N$ and with only one pole in M, but the price to pay is that this pole may be of very high order.

Maslov index. In the previous example, on a closed Riemann surface one obtains a meromorphic function with prescribed zeros by allowing a pole of high order at some point p. If z are holomorphic coordinates near p such that $z(p) = 0$, a typical function with pole of high order is given by $f(z) = z^{-N}$. Now, if one cuts a small set $\{|z| \leq \varepsilon\}$ away from the surface, on the boundary $\{|z| = \varepsilon\}$ the function is given by $f(\varepsilon e^{i\theta}) = \varepsilon^{-N} e^{-iN\theta}$.

The point is that near a pole of high order, a function winds many times around the pole. Similarly, a large winding number over the boundary will ensure the existence of holomorphic functions having prescribed zeros in a manifold with boundary. Since the boundary components of a Riemann surface are diffeomorphic to S^1, we first consider winding numbers on S^1.

We let $S^1 := \mathbb{R}/\mathbb{Z}$ with $\pi : \mathbb{R} \to S^1$, $x \mapsto [x]$ the natural projection. The proof of the next fact is left as an exercise.

Lemma 2.4.2. *(Lifting maps on S^1) Let $f : S^1 \to S^1$ be continuous. There is a continuous map $F : \mathbb{R} \to \mathbb{R}$, unique up to an additive integer constant, such that $\pi \circ F = f \circ \pi$.*

Definition. If $f : S^1 \to S^1$ is continuous, we define the *degree* (or *winding number*) of f by

$$\deg(f) := F(1) - F(0)$$

if F is as in the preceding lemma.

Let now (M, g) be a compact oriented 2D Riemannian manifold with boundary ∂M. We will control the boundary behavior of a function u on M by requiring that $u|_{\partial M}$ is a section of a suitable bundle over the boundary.

Let $E := \partial M \times \mathbb{C}$ be the trivial bundle over ∂M, and let $F \subseteq E$ be a smooth real line bundle. This means that for any $p \in \partial M$, the fiber F_p is a line through the origin in $\mathbb{C}$. Since any such line in $\mathbb{C}$ is equal to $e^{2\pi i\theta}\mathbb{R}$ for some θ, one has a smooth function

$$h_F : \partial M \to S^1, \quad p \mapsto e^{2\pi i\theta(p)}$$

where $F_p = e^{2\pi i\theta(p)}\mathbb{R}$. Now ∂M is a compact oriented 1D manifold, so by the characterization of such manifolds [**6**], for any component $(\partial M)_j$ there is a diffeomorphism $\psi_j : S^1 \to (\partial M)_j$.

Definition. If $F \subseteq E$ is a real line bundle over ∂M, the *Maslov index* of F is

$$\mu(F) := \sum_j \deg(h_F|_{(\partial M)_j} \circ \psi_j).$$

Since any two diffeomorphisms $S^1 \to (\partial M)_j$ are equal up to a diffeomorphism $\psi : S^1 \to S^1$, the degree theory facts that $\deg(\psi) = 1$ and $\deg(f \circ g) = \deg(f)\deg(g)$ ensure that the Maslov index is well defined.

Example. Suppose M is a Riemann surface and $\psi : S^1 \to \partial M$ is a diffeomorphism. Consider a smooth function $\eta : [0,1] \to \mathbb{R}$ such that $\eta(t) = 0$ for $t \leq 1/2 - \varepsilon$ and $\eta(t) = L$ for $t \geq 1/2 + \varepsilon$, for an integer L. Define a real line bundle F over ∂M by

$$F_{\psi(e^{2\pi i\theta})} := e^{2\pi i\eta(\theta)}.$$

Then $h_F(\psi(e^{2\pi i\theta})) = e^{2\pi i\eta(\theta)}$, and the Maslov index is

$$\mu(F) = \deg(h_F \circ \psi) = \eta(1) - \eta(0) = L.$$

This example shows that the Maslov index of F may be arbitrarily large even though F only winds in a very small subset of ∂M.

Riemann-Roch on manifolds with boundary. We are now ready to state the result that we will use. Let $E = \partial M \times \mathbb{C}$ be the trivial bundle over the boundary, and let $F \subseteq E$ be a smooth real line bundle as above. We define the space

$$H^k_F(M) := \{u \in H^k(M)\,;\, u(p) \in F_p \text{ for } p \in \partial M\}$$

and the operator

$$\overline{\partial}_F := \overline{\partial}|_{H^k_F(M)}.$$

The operator $\overline{\partial}_F$ maps $H^k_F(M)$ into the space $H^{k-1}(M; T^*_{(0,1)}M)$ of H^{k-1} sections of $T^*_{(0,1)}M$.

The following result may be found in McDuff-Salamon [**9**].

Theorem 2.4.3. *(Riemann-Roch on manifolds with boundary) Let (M, g) be a compact oriented Riemann surface with boundary ∂M. If*

$$\mu(F) + 2\chi(M) \geq 0,$$

then $\overline{\partial}_F : H^k_F(M) \to H^{k-1}(M; T^*_{(0,1)}M)$ is surjective. Further,

$$\dim \mathrm{Ker}(\overline{\partial}_F) = \chi(M) + \mu(F).$$

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