Inverse problems on Riemann surfaces
Exercises, 04.11.2009
(return by 04.12.2009)

The exercises can be returned to Mikko Salo.

1. Let $F : M \to N$ be smooth and let $p \in M$. Compute a coordinate expression for $F_* X_p \in T_{F(p)} N$ when $X_p \in T_p M$, and for $F^* \eta_{F(p)} \in T_p^* M$ when $\eta_{F(p)} \in T^*_{F(p)} N$.

2. Prove that the local coordinate definition of d on 1-forms is independent of choice of coordinates.

3. Let $M \subseteq \mathbf{R}^3$ be a smooth hypersurface which has an orientation form. Show that there is a smooth map $\hat{n} : M \to S^2$ such that $\hat{n}(p)$ is orthogonal to $T_p M$ for all p.

4. Let ω be a compactly supported 2-form in an oriented 2D manifold M, and define $\int_M \omega := \sum_{j=1}^m \int_{\varphi_j(U_j)} (\varphi_j^{-1})^* (\chi_j \omega)$ where $\{(U_j, \varphi_j)\}$ is a cover of $\mathrm{supp}(\omega)$ by positively oriented charts and $\{\chi_j\}$ is a partition of unity subordinate to $\{U_j\}$. Show that the definition of $\int_M \omega$ is independent of choice of cover and partition of unity.

5. If $p, q \in \mathbf{R}^2$ and g is the Euclidean metric, show that the line segment between p and q is the shortest curve joining these points and $d_g(p, q) = |p - q|$.

6. In (M, g), show that the inner product of cotangent vectors (defined by $g(\omega, \eta) = g(\omega^\sharp, \eta^\sharp))$ is given in local coordinates by $g(\omega_j \, dx^j, \eta_k \, dx^k) = g^{jk} \omega_j \eta_k$ where (g^{jk}) is the matrix inverse of (g_{jk}).

7. Show that in local coordinates, the volume form dV_g of (M, g) is given by $\sqrt{\det(g_{jk})} \, dx^1 \wedge dx^2$.

8. If ω, η are 1-forms show that $*(\omega \wedge *\eta) = g(\omega, \eta)$.

9. If $\Delta_g u := *^{-1} d * du$, prove that in local coordinates
$$\Delta_g u = -\frac{1}{\sqrt{\det(g_{jk})}} \frac{\partial}{\partial x_j} \left(\sqrt{\det(g_{jk})} g^{jk} \frac{\partial u}{\partial x_k} \right).$$

10. If u and v are smooth functions, prove Green's formula

$$\int_M u\Delta_g v\,dV = -\int_{\partial M} u\partial_\nu v\,dS + \int_M \langle du, dv\rangle\,dV.$$

where $\partial_\nu v = \langle dv, \nu\rangle$ and ν is the outer unit normal.

11. Show that compositions and inverses of conformal diffeomorphisms are conformal.

12. (Conformal = angle-preserving) Prove Lemma 2.2.2 in the lectures.

13. Suppose M is a simply connected open manifold, and let ω be a smooth 1-form in M which is closed (that is, $d\omega = 0$). Prove directly that there is a smooth function u in M such that $du = \omega$.

14. (∂ and $\overline{\partial}$ operators) Prove Lemmas 2.3.1, 2.3.2, and 2.3.4 in the lectures.

15. Let $f : S^1 \to S^1$ be continuous. If $F : \mathbf{R} \to \mathbf{R}$ is continuous and $\pi \circ F = f \circ \pi$, show that F is uniquely determined modulo $\mathbf{Z}$ (here $S^1 = \mathbf{R}/\mathbf{Z}$ and $\pi : \mathbf{R} \to S^1$ is the projection).

16. Let (M, g) be a 2D manifold with boundary, let $\Gamma_0 \subseteq \partial M$ be a strict open subset, and let $f \in H^k(\overline{\Gamma_0})$. Show that there is Φ such that $\overline{\partial}\Phi = 0$ in M and $\mathrm{Re}\,(\Phi) = f$ on Γ_0.

17. Let $\Phi = u + iv$ where $u, v : M \to \mathbf{R}$ are smooth and $\overline{\partial}\Phi = 0$. Show that p is a nondegenerate critical point of u iff p is a nondegenerate critical point of v iff $\Phi \approx a + bz^2$ with $b \neq 0$ in any holomorphic coordinate z near p such that $z(p) = 0$.