Elements of Set Theory, Fall 2009

Exercise 9

27.11.2009

Note that the exercise class is in Room B322 this week.

1. Recall: A set C is a *chain*, if for every x and y in C, either $x \subseteq y$ or $y \subseteq x$. Let A and B be sets, and put

$$\mathcal{F} = \{f \subseteq A \times B \mid f \text{ is a 1-1 function } \}.$$

   Show that $\mathcal{F}$ is closed under taking unions of chains, i.e., if $C \subseteq \mathcal{F}$ is a chain, then $\bigcup C \in \mathcal{F}$.

2. Suppose that A is a countable set. Show, *without* the Axiom of Choice, that A has a choice function, i.e., a function $f : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ with $f(x) \in x$ for all $x \in \mathcal{P}(A) \setminus \{\emptyset\}$. *Hint:* Remember that ω has a natural well-ordering.

3. Show that the following statement follows from the Axiom of Choice:

   ($\star$) For any set A there is a function F with $\operatorname{dom}(F) = \bigcup A$, and $x \in F(x) \in A$ for every $x \in \bigcup A$.

   You may use any of the equivalent forms of AC, whichever seems appropriate.

4. Prove the converse of the previous problem, that is, $(\star) \Rightarrow$ AC. Again, you may use any form of AC.

   In the following two exercises you should utilize the theorem that $\kappa \cdot \kappa = \kappa$ for any infinite cardinal κ.

5. Let $\mathcal{P}_{fin}(A)$ be the set of all finite subsets of A. Show that if A is infinite, then $A \approx \mathcal{P}_{fin}(A)$.

6. Generalize the result that a countable union of countable sets is countable: Prove that if $\mathcal{A}$ is a set with the property that $|X| \leq \kappa$ for every $X \in \mathcal{A}$, then

$$|\bigcup \mathcal{A}| \leq |\mathcal{A}| \cdot \kappa.$$