Elements of Set Theory, Fall 2009

Exercise 8

20.11.2009

1. Let κ, λ and μ be cardinal numbers. Prove:
   (a) $\kappa \leq \lambda \Rightarrow \kappa + \mu \leq \lambda + \mu$,
   (b) $\kappa \leq \lambda \Rightarrow \kappa \cdot \mu \leq \lambda \cdot \mu$,
   (c) $\kappa \leq \lambda \Rightarrow \kappa^\mu \leq \lambda^\mu$,
   (d) $\kappa \leq \lambda \Rightarrow \mu^\kappa \leq \mu^\lambda$, if not both κ and μ equal 0.

2. Give counterexamples to show that we cannot replace $\leq$ by $<$ in the above theorem, without using 0. *Hint:* Natural numbers alone cannot give us counterexamples, so consider also $\aleph_0$ and $2^{\aleph_0}$.

3. Show that the following definition gives us a one-to-one function G from ${}^\omega\{0,2\}$ to $\mathbb{R}$. For $f \in {}^\omega\{0,2\}$, let
$$G(f) = \sum_{n\in\omega} \frac{f(n)}{3^{n+1}}.$$
   Here $\sum_{n\in\omega} \frac{f(n)}{3^{n+1}}$ is just the least upper bound for the set $\{\frac{f(0)}{3} + \ldots + \frac{f(n)}{3^{n+1}} \mid n \in \omega\}$. You have to show that G is well defined by showing that the least upper bounds exist, and that G is one-to-one.

   Note: ${}^\omega\{0,2\}$ is of course equinumerous with ${}^\omega 2$, so this gives us the result $2^{\aleph_0} \leq |\mathbb{R}|$. The reason for considering ${}^\omega\{0,2\}$ is that we get a one-to-one function. You might want to draw a picture!

4. Use the following strategy to give an alternative proof for the fact that $\mathbb{R}$ is not equinumerous with ω. Suppose towards a contradiction that $F : \omega \to \mathbb{R}$ was onto. Define recursively reals a_n and b_n, for $n \in \omega$, as follows: $a_0 < F(0)$, b_0 is $F(m)$ with the least m such that $a_0 < F(m) < F(0)$, a_{n+1} is $F(k)$ with the least k such that $a_n < F(k) < b_n$, and b_{n+1} is $F(l)$ with the least l such that $a_{n+1} < F(l) < b_n$. Derive a contradiction by considering the least upper bound for the set $\{a_n \mid n \in \omega\}$.