Elements of Set Theory, Fall 2009

Exercise 6

6.11.2009

1. Is there a function $F : \mathbb{Z} \to \mathbb{Z}$ satisfying $F([\langle m, n\rangle]) = [\langle m, m\rangle]$? How about $G : \mathbb{Q} \to \mathbb{Q}$ satisfying $G([\langle a, b\rangle]) = [\langle a+b, b\rangle]$? If yes, can you explicate the functions in terms of arithmetic in $\mathbb{Z}$ and $\mathbb{Q}$, respectively?

2. Show that for any $r \in \mathbb{Q}$,
$$r <_{\mathbb{Q}} 0_{\mathbb{Q}} \text{ iff } 0_{\mathbb{Q}} <_{\mathbb{Q}} -r.$$

3. Show that the ordering of rationals is *dense*: For any $p, q \in \mathbb{Q}$ with $p <_{\mathbb{Q}} q$, there is an $r \in \mathbb{Q}$ with $p <_{\mathbb{Q}} r <_{\mathbb{Q}} q$.

4. Define the *absolute value* of a real number x by
$$||x|| = x \cup -x.$$
Show that $||x|| \in \mathbb{R}$ and that $||x|| \geq_{\mathbb{R}} 0_{\mathbb{R}}$ for every $x \in \mathbb{R}$.

5. Assume that p is a positive rational number. Show that for any real number x there is a rational number $q \in x$ such that $p + q \notin x$.

6. The multiplication of real numbers is defined as follows:
   (a) If both x and y are nonnegative, then
   $$x \cdot_{\mathbb{R}} y = 0_{\mathbb{R}} \cup \{rs \mid 0 \leq r \in x \text{ and } 0 \leq s \in y\}.$$
   (b) If both x and y are negative, then $x \cdot_{\mathbb{R}} y = ||x|| \cdot_{\mathbb{R}} ||y||$.
   (c) If one of x and y is negative and the other is nonnegative, then $x \cdot_{\mathbb{R}} y = -(||x|| \cdot_{\mathbb{R}} ||y||)$.

   Show that $x \cdot_{\mathbb{R}} y \in \mathbb{R}$ for every $x, y \in \mathbb{R}$. (Note that by the definition it is enough to check this only for nonnegative x and y.)