# Elements of Set Theory, Fall 2009

Exercise 4

16.10.2009

1. Call a natural number *even* if it has the form $2 \cdot m$ for some $m$, and *odd* if it has the form $(2 \cdot p) + 1$ for some $p$. Show that each natural number is either even or odd but never both.

2. Prove that $\omega$ is a transitive set using the well-ordering of $\omega$.

3. Suppose that $A$ is a transitive set. Show that if $\in$ well-orders $A$, i.e., the relation $\{\langle x, y\rangle \in A \times A \mid x \in y\}$ is a well-ordering on $A$, then $\in$ well-orders $A_+$ also.

   In the following exercise we begin using the obvious notational convention of writing $F(x, y)$ instead of $F(\langle x, y\rangle)$. More generally we write $F(x_0, \dots, x_{n-1})$ instead of $F(\langle x_0, \dots, x_{n-1}\rangle)$ when $F : A^n \to B$.

4. Assume that $\sim$ is an equivalence relation on $A$ and that $F : A \times A \to A$. Define that $F$ is *compatible* with $\sim$ if whenever $x \sim x'$ and $y \sim y'$, then also

   $$F(x, y) \sim F(x', y')$$

   Show that there exists a unique function $\bar{F} : (A/\sim) \times (A/\sim) \to (A/\sim)$ s.t.

   $$\bar{F}([x], [y]) = [F(x, y)]$$

   if and only if $F$ is compatible with $\sim$. Note that this is an analogue for Theorem 3Q in the book, which we proved in the class.

5. The *multiplication* of integers is defined as follows:

   $$[\langle m, n\rangle] \cdot_{\mathbb{Z}} [\langle p, q\rangle] = [\langle mp + nq, mq + np\rangle].$$

   Show that this definition is well posed, that is, the function $\cdot_{\mathbb{Z}} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ exists and is unique (use the previous exercise).