Elements of Set Theory, Fall 2009

Exercise 3

9.10.2009

1. Assume that $A \times B = C \times D$. Under what conditions we can conclude that $A = C$ and $B = D$?

2. Suppose that $\bigcup(a_+) = a$. Show that a is $\in$-transitive. (The converse was shown in the class.)

3. Show that a is $\in$-transitive if and only if $\mathcal{P}(a)$ is $\in$-transitive.

4. Prove the *associative law* for natural numbers: For all m, n, $p \in \omega$,

$$m + (n + p) = (m + n) + p.$$

   Hint: induction on p.

5. Let $F : A \to A$ be a one-to-one function, and suppose that $c \in A \setminus \operatorname{ran}(F)$. Define $h : \omega \to A$ recursively by

$$h(0) = c,$$

$$h(n_+) = F(h(n)).$$

   Show that h is one-to-one. (Hint: induction. This was essentially a part of the proof that any Peano system is isomorphic to ω.)

6. Suppose that $f : B \to B$ is a function, and $A \subseteq B$. We have two alternative methods for defining the *closure* of A under f:

   - The "downwards" method: Let $C^* = \bigcap\{X \in \mathcal{P}(B) \mid A \subseteq X \wedge f[X] \subseteq X\}$.
   - The "upwards" method: Let $h : \omega \to \mathcal{P}(B)$ be the function defined recursively by

$$h(0) = A,$$

$$h(n_+) = h(n) \cup f[h(n)],$$

     and define $C_* = \bigcup_{n\in\omega} h(n)$.

   Show that these definitions yield the same result, that is, $C^* = C_*$.