Elements of Set Theory, Fall 2009

Exercise 2

2.10.2009

1. Prove the following extensionality principle for functions: If F and G are functions for which $\operatorname{dom}(F) = \operatorname{dom}(G)$ and $F(x) = G(x)$ for every $x \in \operatorname{dom}(F)$, then $F = G$.

2. Show that if F and G are functions, then $F \cap G$ is a function. Give examples of functions F and G so that $F \cup G$ is not a function.

3. If F is any set, show that

   (a) $F\left[\bigcup \mathcal{A}\right] = \bigcup\{F[A] \mid A \in \mathcal{A}\}$,

   (b) $F\left[\bigcap \mathcal{A}\right] \subseteq \bigcap\{F[A] \mid A \in \mathcal{A}\}$ and equality holds if F is single-rooted.

   (This is Theorem 3K in Enderton's. You can see the book for partial solutions. For (b), remember that we defined $\bigcap \emptyset = \emptyset$.)

4. Show that the following versions of Axiom of Choice are equivalent:

   (AC1) For every function F such that $F(x) \neq \emptyset$ for every $x \in \operatorname{dom}(F)$, there exists a function f with $\operatorname{dom}(f) = \operatorname{dom}(F)$ and $f(x) \in F(x)$.

   (AC2) For every relation R there exists a function $H \subseteq R$ such that $\operatorname{dom}(H) = \operatorname{dom}(R)$.

   We already proved (AC1)⇒(AC2) in the class, so you only need to prove (AC2)⇒(AC1).

5. We proved in the class that if R is an equivalence relation on A, then A/R is a partition of A. Formulate and prove the other direction of this result, i.e., if Π is a partition of A, define an equivalence relation R on A so that $\Pi = A/R$.

6. Suppose that R is a linear ordering. Show that R^{-1} is a linear ordering.