Elements of Set Theory, Fall 2009

Exercise 12

18.12.2009

The following two problems, that give a brief view to somewhat larger cardinal numbers than those we have considered so far, will not be considered in the exercise bonuses. However, you can gain some extra points by solving them. For those who took the exam on 15.12., you can also ask about the exam during this last exercise session.

1. Show that there is an ordinal number α with $\alpha = \aleph_\alpha$. *Hint:* Let recursively $\alpha_0 = \aleph_0$, and $\alpha_{n_+} = \aleph_{\alpha_n}$, and then $\alpha = \bigcup_{n \in \omega} \alpha_n$.

2. A cardinal κ is called *inaccessible* if it is a limit cardinal (it is not of the form $\aleph_{\alpha_+}$ for any α), and there is no set $A \subseteq \kappa$ with $|A| < \kappa$ such that

$$\forall \alpha < \kappa \exists \beta \in A (\alpha \leq \beta).$$

Show that if κ is inaccessible, then necessarily $\kappa = \aleph_\kappa$, but the least cardinal satisfying this is not inaccessible.