Elements of Set Theory, Fall 2009

Exercise 11

11.12.2009

1. Show that the natural ordering of ordinal numbers is defined by proper inclusion as well as by membership, that is,

$$\alpha \in \beta \text{ iff } \alpha \subsetneq \beta.$$

2. Show that
   (a) any ordinal number α is the collection of all smaller ordinal numbers,
   (b) $\alpha_+ = \alpha \cup \{\alpha\}$ is the least ordinal number greater than α,
   (c) if A is a set of ordinals, then $\bigcup A$ is the least ordinal number greater or equal to every element of A.

3. Assume that $\langle A, < \rangle$ is a well-ordered structure, and $t \in A$. Show that $\langle \text{seg}\,(t), < \rangle$ is not isomorphic to $\langle A, < \rangle$.

   Conclude that for well-ordered structures $\langle A, <_A \rangle$ and $\langle B, <_B \rangle$, *exactly* one of the following holds: $\langle A, <_A \rangle \cong \langle B, <_B \rangle$, or $\langle A, <_A \rangle$ is isomorphic to some initial segment of $\langle B, <_B \rangle$, or $\langle B, <_B \rangle$ is isomorphic to some inial segment of $\langle A, <_A \rangle$

4. Suppose that α and β are ordinals. Show that $\alpha \in \beta$ if and only if $\alpha_+ \in \beta_+$. *Hint:* trichotomy.

5. An ordinal α is a *limit ordinal*, if $\alpha \neq \beta_+$ for any ordinal β. Show that an ordinal α is a limit if and only if $\alpha = \bigcup \alpha$.

6. Prove the following version of transfinite induction: Suppose that $\phi(x)$ is any formula such that for every ordinal α, if $\phi(\beta)$ holds for all $\beta \in \alpha$, then also $\phi(\alpha)$ holds. Then $\phi(\alpha)$ holds for every ordinal α.