Elements of Set Theory, Fall 2009

Exercise 10

4.12.2009

1. Define a relation $<_L$ on $\omega \times \omega$ by

$$\langle m, n \rangle <_L \langle m', n' \rangle \text{ iff } m < m' \text{ or } (m = m' \text{ and } n < n').$$

   Show that $<_L$ is a well-ordering. ($<_L$ is called the *lexicographic* ordering of $\omega \times \omega$.)

2. Show that there is no function f from ω *onto* $\omega \times \omega$ with the property that

$$m < n \Rightarrow f(m) <_L f(n).$$

3. By an earlier exercise, we know that $\omega_+ = \omega \cup \{\omega\}$ is well-ordered by $\in$. Show that there is no function f from ω onto ω_+ with the property that

$$m < n \Rightarrow f(m) \in f(n).$$

4. Assume that $<$ is a well-ordering on A, and $f : A \to A$ is an increasing function, that is,

$$x < y \Rightarrow f(x) < f(y).$$

   Show that $x \leq f(x)$ for every $x \in A$. *Hint:* Consider $f(f(x))$ for the least counterexample x.

5. Assume that $S \subseteq \mathbb{R}$ is well-ordered by the usual ordering of reals. Show that S is countable. *Hint:* Choose for each $x \in S$ a rational q such that q is strictly between x and the next element of S. Do you need to use AC?

6. Suppose that R is a well-ordering of A. Show that if R^{-1} also well-orders A, then A is finite.