STOCHASTIC POPULATION MODELS

EXERCISES 10-15

10.
Consider the Wiener process $\{W(t)\}_{r\geq 0}$ with $W(0)=0$. Show that $\mathcal{E}\{W(t)W(s)\} = \min\{t,s\}$

11.
Integrate the linear stochastic differential equation $dX = g(t)dW$ with $X(0) = x_0$. Show that it doesn't matter how we sample the integrant, and so, in particular, it doesn't matter whether we interpret the differential equation as an Ito equation or a Stratonovitch equation.

12.
Integrate the following stochastic differential equations with deterministic initial condition $X(0) = x_0$ for suitable choice of x_0:

(a) $dX = X^2\, dW$ (S)

(b) $dX = e^{-t} X\, dW$ (S)

(c) $dX = -X\, dt + X\, dW$ (I)

13.
Given the stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ and the constants a and b, define $Z(t) := aX(t) + bY(t)$, and show that the auto-covariance function of Z is $C_{Z,Z} = a^2 C_{X,X} + abC_{X,Y} + abC_{Y,X} + b^2 C_{Y,Y}$.

14.
Given the stochastic processes $\{X(t)\}$ and $\{Y(t)\}$ show that Moreover, one has

(a) $C_{\frac{dX}{dt},Y} = +C'_{X,Y}$

(b) $C_{X,\frac{dY}{dt}} = -C'_{X,Y}$

(c) $C_{\frac{dX}{dt},\frac{dY}{dt}} = -C''_{X,Y}$

where $C'_{X,Y}$ and $C''_{X,Y}$ are the first- and second-order derivatives of the cross-covariance function $C_{X,Y}(\tau)$ with respect to τ.

15.
Calculate the spectral density and (if possible) the auto-covariance of $\theta(t)$ defined by the following equations:

(a) $d\theta = -a\theta\, dt + b\, dW$

(b) $d\theta = -a\theta_{\Delta t}\, dt + b\, dW$ where $\theta_{\Delta t}(t) := \theta(t - \Delta t)$

(c) $\theta = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} dW$

(d) $\theta = a \int_{-\infty}^{t} e^{-a(t-s)} \xi(s) ds$ where $\xi(t)$ is the Gaussian white noise

(e) $\theta = \frac{1}{\Delta t} \int_{t-\Delta t}^{t} \eta(s) ds$ where $d\eta = -\eta\, dt + dW$

for positive a and b.