Homework exercises , set 8

1. Give the details of the construction of the direct sum vector bundle $E \oplus F$ when E, F are vector bundles over the same manifold M; the fiber $(E \oplus F)_x$ is $E_x \oplus F_x$. Show that there is a real rank one vector bundle E over S^2 such that $TS^2 \oplus E$ is trivial.

2. Let L be real rank one vector bundle over S^1. Show that $L \oplus L$ is trivial.

3. Construct a connection in the Hopf bundle $S^3 \to S^7 \to S^4$.

4. Let $E \to M$ be a rank n vector bundle and assume that there is a section $\psi \in \Gamma(E)$ such that $\nabla_X \psi = 0$ for all $X \in D^1(M)$ and $\psi(x) \neq 0$ for all $x \in M$. Show that the curvature form F of the vector bundle can be thought of as a $(n-1)\times(n-1)$ matrix valued 2-form. We assume that the structure group is reduced to $U(n)$ or $O(n)$ and $|\psi(x)| = 1$.

5. Think of the group $SU(2)$ as a principal $\mathbb{Z}_2$ bundle over the rotation group $SO(3)$. Determine a connection and its connection form in this bundle.

6. Let $n > 1$, $f : S^{2n-1} \to S^n$ a smooth map and Ω_n a closed $n-$ form on S^n such that $\int_{S^n} \Omega_n = 1$. By cohomological reasons $f^*\Omega_n$ is exact, $f^*\Omega_n = d\omega_{n-1}$.

a) Show that

$$H(f) = \int_{S^{2n-1}} \omega_{n-1} \wedge d\omega_{n-1}$$

does not depend on the choice of ω_{n-1}.

b) Show that $H(f) = H(g)$ when f, g are homotopic.

c) Show that $H(f) = 0$ when n is odd.

d) Compute $H(f)$ when f is the Hopf projection $S^3 \to S^2$.