Homework exercises, set 7

1. Construct the transition functions for the Hopf fibration $SU(2) \to S^2 = SU(2)/U(1)$.

2. Suppose P is a principal bundle over the 4-sphere S^4 given in terms of transition functions as follows. Write S^4 as an union of two extended open half spheres; the intersection is an open interval times the 3-sphere (equator) S^3. Identifying S^3 as the group $G = SU(2)$ we define the transition function on the equator as the identity map $SU(2) \to SU(2)$ (constant on the open interval). Show that this bundle cannot be trivial.

3. Prove the Bianchi identity $dF + [A, F] = 0$ for the connection and curvature in a principal bundle.

4. Think of the tangent bundle of the unit sphere as an associated vector bundle to a principal bundle $P \to S^2$ with fiber $U(1)$. Explain the connection on P, in terms of a horizontal distribution, which corresponds to the Levi-Civita connection on S^2 with standard metric.

5. Let P be a principal G bundle over M and $f : N \to M$ a smooth map. Define the pull-back bundle f^*P over N such that the fiber at $x \in N$ is identified as the fiber of P at $f(x)$. Show that f^*P is indeed a smooth principal bundle over N with structure group G. Let $\pi : P \to M$ be the projection. Show that π^*P over P is trivial.

6. Construct the transition functions for the Hopf fibration $S^7 \to S^7/SU(2) = S^4$. Here S^7 is the unit sphere in $\mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2$ and the right action of $g \in SU(2)$ is given as the natural action on the (row) vectors in both summands $\mathbb{C}^2$. Hint: Show that $S^7/SU(2) = S^4$ by first observing that the $SU(2)$ orbits in $S^7 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$ (with $(w_1, w_2, z_1, z_2) \in S^7$) are uniquely parametrized by (w_1, w_2) when $|z_1|^2 + |z_2|^2 \geq 1/2$ and by (z_1, z_2) when $|w_1|^2 + |w_2|^2 \geq 1/2$. Both of these sets can be identified as 4-dimensional half-spheres $S^4_{\pm}$.