Homework exercises set 6

1. Prove $ad_g([X,Y]) = [ad_g(X), ad_g(Y)]$ on any Lie group.

2. Show that the Lie algebra of left invariant vector fields on a matrix Lie group is isomorphic to the Lie algebra of right invariant vector fields.

3. Locally, near the neutral element in a Lie group, one can write $\exp(X) \cdot \exp(Y) = \exp(Z)$. Show that in the case of a matrix group one has $Z = X + Y + \frac{1}{2}[X,Y]$ + higher order terms.

4. Let G be the group of real $n \times n$ matrices with zeros under the diagonal and one's on the diagonal. Show that the exponential function $\exp : Lie(G) \to G$ is a diffeomorphism.

5. In the case of the group $SU(2)$ one can use the parametrization $g = g(x) = e^{i\sigma \cdot x}$ for the group elements. Close to the unit element this is local chart on $SU(2)$. Here σ_i with $i = 1,2,3$ is a basis of traceless hermitean matrices (the Pauli matrices) and $x \in \mathbb{R}^3$. Derive an explicit formula for the adjoint action $\sigma \cdot y \mapsto g(x)\sigma \cdot y g(x)^{-1}$.

6. Show that the group $SL(2,\mathbb{R})$ of real 2×2 matrices with det $= 1$ is diffeomorphic to the space $S^1 \times \mathbb{R}^2$. Show that the exponential map is not onto. Show that the Lie algebra of $SL(2,\mathbb{R})$ is isomorphic to the Lie algebra of $SO(2,1)$. Here $SO(2,1)$ is the group of linear transformations in $\mathbb{R}^3$ which preserve the pseudonorm $|x|^2 = x_1^2 + x_2^2 - x_3^2$ and have determinant =1. The Lie algebra is determined in the same way as in the case of $SO(n)$: The group consists of matrices A such that $\det(A) = 1$ and $A^t g A = g$. For $SO(n)$ $g = 1$ whereas for $SO(2,1)$ $g = \mathrm{diag}(1,1,-1)$.