

Homework exercises set 5

1. Compute the curvature tensor of the unit sphere  $S^2$  from the Riemann metric induced by the euclidean metric in  $\mathbb{R}^3$ . Hint: The only independent components (in spherical coordinates) are  $R_{\phi\theta\phi}^\theta$  and  $R_{\theta\theta\phi}^\phi$  since  $R$  is antisymmetric in the last two of the lower indices.

2. Prove the relation  $\sum_i \Gamma_{ij}^i = \frac{1}{2} \partial_j \log \det(g)$  for the Levi-Civita connection.

3. It is sometimes useful to compute connection and curvature in a local basis different from a coordinate basis. That is, we can define Christoffel symbols

$$\nabla_a e_b = \sum_c \Gamma_{ab}^c e_c,$$

where  $e_1, e_2, \dots, e_n$  is an arbitrary local basis of vector fields and  $\nabla_a = \nabla_{e_a}$ . Relate these Christoffel symbols to the coordinate basis Christoffel symbols  $\Gamma_{ij}^k$ . Let next  $e_1, e_2$  be a local orthonormal basis of vector fields on  $S^2$ . Compute the curvature tensor in this basis. Compute also the curvature tensor on a sphere of radius  $r$ .

4. Show directly from the definition of parallel transport that the length of a parallel vector field along a curve is constant, in the case of Levi-Civita connection.

5. On a manifold  $M$  we have a global basis of orthonormal vector fields  $L_1, L_2, L_3$  with nonzero commutation relations  $[L_1, L_2] = L_3$ ,  $[L_2, L_3] = L_1$ , and  $[L_3, L_1] = L_2$ . Compute the Christoffel symbols for the Levi-Civita connection. Hint: Levi-Civita connection is the unique metric compatible torsion free connection on  $M$ . Use repeatedly the symmetry properties of Christoffel symbols.

6. A ship starts from a position in the Atlantic Ocean with coordinates  $10^\circ$  North  $30^\circ$  West (Cape Verde Islands). It sails directly North to the  $45^\circ$  Northern latitude (Azores, Portugal) and then turns abruptly to the West and sails until it hits the  $60^\circ$  Western longitude (Nova Scotia, Canada). Suppose a vector is parallel transported along this route (with the help of a gyroscope). Its initial direction is  $45^\circ$  (NE). What is its final direction?