PRINCIPAL BUNDLES AND YANG-MILLS THEORY

Homework exercises set 4

1. Let M be a pseudo-Riemannian manifold of signature (p, q), that is, the metric has p positive and q negative eigenvalues. Let $\omega \in \Omega^k(M)$. Show that $**\omega = \pm\omega$ and compute the phase $\pm$ as a function of p, q, and k.

2. Show that $f^*(d\omega) = d(f^*\omega)$ for any form ω on a manifold N and a smooth map $f : M \to N$.

3. Show that the definition of the integral of $\omega \in \Omega^n(M)$ over a n-dimensional manifold M does not depend on the choice of a partition of unity.

4. We define Christoffel symbols on the unit sphere S^2 in terms of spherical coordinates (θ, ϕ), away from the poles $\theta = 0, \pi$, as

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2}\sin 2\theta \text{ and } \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

and all the other symbols equal to zero. Show that there is a globally defined covariant differentiation ∇ corresponding to these Christoffel symbols.

5. Show that the equations

$$\frac{dY^i(s)}{ds} + \Gamma^i_{jk}(x(s))\frac{dx^j(s)}{ds}Y^k(x(s)) = 0$$

for parallel transport are consistent with coordinate transformations.

6. Compute the de Rham cohomology of the torus $T^2 = S^1 \times S^1$ using the Mayer-Vietoris sequence.