Homework exercises set 3

1. Let $t \mapsto h_t$ be the (local) flow generated by a vector field X. Let Y be another vector field and define

$$(\mathcal{L}_X Y)(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[(h_{-\epsilon})_* \cdot Y(h_\epsilon(x)) - Y(x) \right].$$

Show that $\mathcal{L}_X(Y) = [X, Y]$.

2. Show that $f_*[X, Y] = [f_* X, f_* Y]$.

3. Let X, Y be a pair of vector fields and ω, η differential forms on a manifold. Show that $i_{[X,Y]}\omega = \mathcal{L}_X(i_Y \omega) - i_Y(\mathcal{L}_X \omega)$ and that $i_X(\omega \wedge \eta) = i_X \omega \wedge \eta + (-1)^k \omega \wedge i_X \eta$ when ω is a k-form. Furthermore, observe that $i_X^2 = 0$ and $\mathcal{L}_X i_X = i_X \mathcal{L}_X$.

4. Let $M = \mathbb{R}^2 \setminus \{0\}$ and $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$. Show that ω is closed. Define $f(x, y) = \text{arc } \tan(y/x)$. Show that $\omega = df$. Is ω exact?

5. Prove that the relation

$$\begin{aligned}(d\omega)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i-1} X_i \cdot \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})\end{aligned}$$

is compatible with the definition of d in terms of local coordinates. Here the hat means that the corresponding term is deleted.

6. Show directly, without using general theorems on cohomology of product spaces, that $H^2(S^1 \times S^1) = \mathbb{R}$.