Homework exercises, set 11

1. Denote by γ_μ, $\mu = 0,1,2,3$, the Dirac matrices in the Minkowski metric $\eta = \mathrm{diag}(1,-1,-1,-1)$ where the index $\mu = 0$ corresponds to the time-like (positive norm) direction. Compute the commutation relations of the matrices $M_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ and show that the Lie algebra generated by these matrices is isomorphic to the Lie algebra of complex trace-less 2×2 matrices, considered as a *real* vector space.

2. Let A be a local vector potential, that is, the pull-back to the base of the connection form ω on the the total space P of a principal bundle. Assume that A is defined on an open set $U \subset M$ in the given local trivialization. What is the horizontal subspace $H_p \subset T_pP$ expressed in the local trivialization $U \times G$ of the principal bundle?

3. Let $\{U_\alpha\}$ be an open cover of a compact oriented manifold M and $\{\rho_\alpha\}$ a partition of unity subordinate to this open cover. Assume that we have a collection of integers $n_{\alpha\beta\gamma}$, antisymmetric in the indices, corresponding to the labels of the open cover, such that

$$n_{\alpha\beta\gamma} - n_{\alpha\beta\delta} + n_{\alpha\gamma\delta} - n_{\beta\gamma\delta} = 0$$

for all quadruples of indices. Show that the collection of local 2-forms

$$F_\alpha = \sum_{\beta\gamma} n_{\alpha\beta\gamma} d\rho_\beta \wedge d\rho_\gamma$$

actually defines a globally well-defined closed 2-form on M. What has this to do with the first Chern class of a complex line bundle, or its transition functions?

4. Write down explicitly the Dirac operator on the unit sphere S^2, coupled to the vector potential of the monopole bundle.

5. a) Compute the index of the Dirac operator on S^4 (standard metric) associated to a principal $SU(2)$ bundle through the defining representation of $SU(2)$ in $\mathbb{C}^2$. We assume that the connection corresponds to the basic instanton (self-dual) gauge field, the value of the integral of (the normalized) second Chern class is equal to one. Use the fact that the first Pontrjagin number of S^4 vanishes (it is actually not very difficult to prove this). b) Construct a Dirac operator in a vector bundle over the torus $S^1 \times S^1$ with a given index $= k$.