Homework exercises, set 10

1. Prove the commutation relations

$$[M_{ij}, M_{kl}] = \delta_{jk} M_{il} + \delta_{il} M_{jk} - \delta_{ik} M_{jl} - \delta_{jl} M_{ik}.$$

for the matrices $M_{ij} = \frac{1}{4}[\gamma_i, \gamma_j]$. (Euclidean metric)

2. Show that the unit sphere in any dimension is a spin manifold. Hint: Think of $Spin(n)$ as a $\mathbb{Z}_2$ principal bundle over $SO(n)$. The transition function of the frame bundle of S^n defines by pull-back a Z_2 bundle over the equator.

3. Compute the 8-form part of the Chern character tr exp $(F/2\pi)$ of a real vector bundle in terms of the Pontrjagin classes.

4. We identify the Euclidean space $\mathbb{R}^4$ as the quaternion algebra, i.e., the algebra of complex 2×2 matrices $q = x_4 \cdot \mathbf{1} + \mathbf{x}$, where x_4 is a real number and $\mathbf{x}$ is a traceless antihermitean matrix. The traceless part $\mathbf{x}$ can also be interpreted as an element in the Lie algebra of the group $SU(2)$. We denote $\mathrm{Im}(q) = \mathbf{x}$ and $|q|$ is the Euclidean norm of q.

Define a Lie algebra valued 1-form as $A = \frac{1}{|q|^2+1}\mathrm{Im}(q dq^*)$, where q^* is the hermitean conjugate of the matrix q. We interprete this as a connection form on a $SU(2)$ bundle over $\mathbb{R}^4$. Show that the curvature form is

$$F = \frac{1}{(|q|^2+1)^2} dq \wedge dq^*.$$

Show that F is self-dual and that its instanton number is equal to 1. Remark: Although all bundles over $\mathbb{R}^4$ are trivial, it is not difficult to show that the form A actually comes from a connection of a principal $SU(2)$ bundle over S^4, as pull-back through stereographic projection. This is the Belavin-Polyakov-Schwarz-Tyupkin instanton (BPST).

5. The construction in Problem 4 can be generalized to arbitrary instanton number k. Read the article ADHM: Atiyah, Drinfeld, Hitchin, Manin: Construction of instantons, Phys. Lett. A , vol. 65 (1978), p. 185-187. Explain the case $k = 2$ and $G = SU(2)$.