Homework exercises, set 1

1. Let M be the 2-dimensional torus $S^1 \times S^1$. Construct a differentiable structure on M using an atlas consisting of two open sets.

2. The standard spherical coordinates (θ, ϕ), with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, on the unit sphere S^2 do not suffice to define a differentiable structure. (Why?) Find a 'minimal modification', in terms of two coordinates charts, to make S^2 to a manifold.

3. The group $SL(2, \mathbb{R})$ of real 2×2 matrices with determinant equal to 1 is a manifold. How?

4. The unit sphere S^3 can be thought of as the group $SU(2)$ of unitary complex 2×2 matrices with determinant $= 1$. Using this fact show that the tangent bundle TS^3 can be identified as the Cartesian product $\mathbb{R}^3 \times S^3$.

5. Check the relations

$$[X, fY] = f[X, Y] + (X \cdot f)Y \text{ and } [fX, Y] = f[X, Y] - (Y \cdot f)X$$

for a smooth function f and a pair of vector fields X, Y on a manifold.

6. Let M be the manifold of real nonsingular $n \times n$ matrices. For each real $n \times n$ matrix X we define a flow h^X on this manifold by $h_t^X(g) = e^{-tX} g$, with ordinary matrix multiplication. This flow defines a vector field $\hat{X}$ on M as usual and for a smooth function f on M

$$(\hat{X}.f)(g) = \frac{d}{dt} f(h_t^X(g))|_{t=0}.$$

Show that the commutator $[\hat{X}, \hat{Y}]$ of vector fields corresponds to the commutator of matrices $[X, Y]$, i.e. $[\hat{X}, \hat{Y}] = \widehat{[X, Y]}$.