CHAPTER 6 THE CHIRAL ANOMALY

6.0. Introduction

The Dirac operator on a manifold M is a first order partial differential operator acting on sections of a spin bundle over M. The Dirac operator is elliptic when the metric of M is positive definite. The main task in this chapter is to study properties of the determinant of the Dirac operator.

The space of sections of the spin bundle is infinite-dimensional. The determinant of a linear operator in a Hilbert space is a priori well-defined only if it is of type 1 + a trace-class operator. However, the Dirac operator is never of this type. In order to define the determinant one must "regularize" the Dirac operator. There is a great freedom in choosing the regularization; the requirement is that the regularized determinant should display the essential information about the spectrum of the original operator (especially the zeros of the operator) and should be continuous in the possible parameters.

We shall study the case when the Dirac operator $D = D(A)$ is parametrized by a vector potential A. The Dirac equation transforms equivariantly with respect to gauge transformations and therefore one would expect that the determinant $\det D(A)$ is gauge invariant. This is indeed the case when D operates on standard Dirac fermions consisting of components of both *chiralities* $\pm$. However, when the Dirac field is massless the components belonging to the opposite chiralities decouple and it is natural to study D_+ and D_- separately. (D_+ is the *Weyl-Dirac* operator which maps positive chirality spinors to negative chirality spinors and D_- goes to the opposite direction.) In this case it turns out that one cannot regularize the determinant in a gauge invariant way but there is a *chiral anomaly*, which measures how the determinant is changed under a gauge transformation.

The anomaly of the Dirac determinant manifests itself as a source for the *axial vector current* . In the same way as the electromagnetic current is associated to gauge tranformations of the electromagnetic vector potential, the gauge transformations acting on Dirac field through a phase transformation, there is a chiral current j^5_μ such that the time component j^5_0 generates the chiral rotations (phase transformations, opposite phases for chiralities $\pm$). Classically, in the case of a massless Dirac field, the chiral current is conserved, $\partial^\mu j^5_\mu = 0$. However, in the case of chiral fermions (only half of the fermion components are coupled to the vector potential) there is an anomaly: The divergence of the chiral current does not vanish. This was the way anomalies were originally found (in perturbation theory computations of the chiral current) [Adler, 1969; Bardeen, 1969; Bell and Jackiw, 1969; Brown, Shi, and Young, 1969].

When the gauge group is non-Abelian there is a similar phenomenon; there is an additional Lie algebra index labelling the different components of the current and classically the divergence equation is replaced by the covariant divergence $\nabla^\mu j^5_\mu = 0$, where $\nabla^\mu = \partial^\mu + [A^\mu, \cdot]$ is the covariant derivative defined by the gauge potential A_μ [Gross and Jackiw, 1972].

It was realized much later that there is geometrical and topological reason for the occurrence of anomalies in the "effective action" (=logarithm of the determinant of the Dirac operator). The anomalies can be derived using Atiyah-Singer index theory. The index of a Dirac operator D is the difference $n_+(D) - n_-(D)$ of the multiplicity n_+ of the zero eigenvalue of D in the positive chirality sector and the number n_- of negative chirality zero modes. According to the Atiyah-Singer index theory that number can be expressed as an integral of certain characteristic class (involving the Chern classes) over the space-time manifold M. The density under the integral is the divergence of the (Abelian) axial vector current [Nielsen, Römer, and Schroer, 1977, 1978; Nielsen and Schroer, 1978; Jackiw and Rebbi, 1977].

In the case of non-Abelian chiral transformations one has to use *families index theory*. Again, characteristic classes are involved. The anomaly can be neatly expressed through the curvature of the space $\mathcal{A}/\mathcal{G}$ of vector potentials modulo gauge transformations. We shall explain this point of view in Section 5.4, following closely the presentation in Atiyah and Singer [1984]; see also Alvarez-Gaume and Ginsparg [1984].

The anomalies manifest themselves also in the Hamiltonian approach in a variety of ways. There is a relation between anomalies and pair production of particles [Alvarez-Gaume and Ginsparg, 1984]. For the main theme for this book the most important consequence of anomalies is the fact that the current algebra will be modified: There are Schwinger terms in the commutators. These commutators are precisely those which we have already studied in the previous chapter!

One can view the chiral anomalies also as the noninvariance of the fermionic path integral measure under gauge transformations; we shall not discuss that point of view here; see Fujikawa [1979].

Our treatment of anomalies will use cohomological methods. However, there are many aspects of this approach which we cannot cover in this book; for more specialized discussions see, e.g., Bonora and Cotta-Ramusino [1983]; Bonora, Cotta-Ramusino, Rinaldi, and Stasheff [1987, 1988], and references therein.

6.1. The Clifford algebra

Let H be a real vector space equipped with an inner product $(\cdot,\cdot)$. The *Clifford algebra* $C(H)$ based on this data is the associative algebra containing the identity 1 and generated by the vectors $x \in H$ subject to the defining relations

$$xy + yx = 2(x,y). \tag{6.1.1}$$

Assume in the following that H is finite-dimensional with an orthonormal basis $\{e_1, \dots e_n\}$. A basis for the algebra $C(H)$ is given by 1 and the products

$$e_{i_1} e_{i_2} \dots e_{i_p},\ 1 \le i_1 < i_2 < \dots < i_p \le n \tag{6.1.2}$$

since $e_i^2 = 1$ and $e_i e_j = -e_j e_i$ for $i \neq j$. Thus the dimension of $C(H)$ is $\sum_{p=0}^{n} \binom{n}{p} = 2^n$.

A (reducible) representation of $C(H)$ can be constructed in the vector space $\Lambda(H)$, in the exterior algebra of H, as follows. Denote $dx = x\wedge$, the exterior multiplication by the vector x, and $\imath_x$ the contraction operator defined by linearity and

$$\imath_x(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} (x, e_{i_k}) e_{i_1} \wedge \ldots \hat{e}_{i_k} \cdots \wedge e_{i_p}$$

where the caret means that e_{i_k} has been deleted. The basic commutation relations are

$$dx\, dy = -dy\, dx,\ \imath_x \imath_y = -\imath_y \imath_x,\ dx\, \imath_y + \imath_y\, dx = (x, y). \tag{6.1.3}$$

A representation of $C(H)$ is obtained as $x \mapsto \gamma(x) = dx + \imath_x$. The dimension of the representation is equal to $\dim\Lambda(H)$=$\dim C(H) = 2^n$.

Next we shall consider the case when n is even, $n = 2m$. In order to define an irreducible representation of $C(H)$ we shall shall complexify the Clifford algebra, $C(H)_{\mathbb{C}} = C(H) \underset{\mathbb{R}}{\otimes} \mathbb{C}$. Let us define

$$a_k = \frac{1}{2}(e_k + ie_{k+m}),\ a_k^* = \frac{1}{2}(e_k - ie_{k+m}), k = 1, 2, \ldots, m.$$

They satisfy the canonical anticommutation relations

$$[a_i^*, a_j]_+ = \delta_{ij} \tag{6.1.4}$$

and all other anticommutators =0. We have

$$e_k = a_k + a_k^*,\ e_{k+m} = i(a_k^* - a_k) \qquad 1 \leq k \leq m. \tag{6.1.5.}$$

The fermionic Fock space $\mathcal{F}$ is by definition the complex Clifford algebra modulo the left ideal generated by the *annihilation operators* a_k. A basis of $\mathcal{F}$ consists of the vectors $a_{i_1}^* \ldots a_{i_p}^* \cdot 1$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and the vector $1 \in C(H)_{\mathbb{C}}$ is *the vacuum* in $\mathcal{F}$. The dimension of the Fock space is 2^m. By (6.1.5) the Fock space carries a representation of the Clifford algebra; we denote by γ_i the operator representing e_i in $\mathcal{F}$. It is a simple exercise to show that this representation is irreducible.

Define $\gamma_{2m+1} = -i^m \gamma_1 \ldots \gamma_n$, called the *chirality operator*. From the anticommutation relations $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$ follows that γ_{2m+1} anticommutes with each γ_i, $1 \leq i \leq n$, and $\gamma_{2m+1}^2 = 1$. It follows that we can construct in the same Fock space $\mathcal{F}$ a representation of the Clifford algebra based on the *odd dimensional* vector space $H \oplus \mathbb{R}e_{2m+1}$ by representing e_{2m+1} by the operator γ_{2m+1}.

Orthogonal transformations of H extend to automorphisms of the Clifford algebra, as can be seen from the defining relations (6.1.1). In a given complex representation γ of $C(H)$ in a vector space V we would like to represent the automorphisms $R \in SO(n)$ by linear operators $T(R)$. Let us first consider the even dimensional case, $n = 2m$. A complete basis of the Lie algebra $\mathbf{so}(n)$ is given by the matrices $s_{ij} = e_{ij} - e_{ji}$, $1 \leq i < j \leq n$, where the e_{ij}'s are elements of the Weyl basis of the general linear algebra $\mathbf{gl}(n)$ with commutation relations $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$. Set

$$T(s_{ij}) = \frac{1}{2}\gamma(e_i)\gamma(e_j) = -T(s_{ji}).$$

Then, by a straightforward computation,

$$[T(s_{ij}), T(s_{kl})] = \delta_{jk}T(s_{il}) + \delta_{il}T(s_{jk}) - \delta_{ik}T(s_{jl}) - \delta_{jl}T(s_{ik}) \tag{6.1.6}$$

which are precisely the commutation relations of the matrices s_{ij}. Thus we have a representation of the Lie algebra $\mathbf{so}(n)$ in V. Furthermore,

$$[T(s_{ij}), \gamma(e_k)] = \delta_{jk}\gamma(e_i) - \delta_{ik}\gamma(e_j) \tag{6.1.7}$$

which shows that the generators $\gamma(x)$ transform like vectors under the adjoint action of the representation T of the Lie algebra of $SO(n)$. The question is now: can we exponentiate the infinitesimal generators $T(s_{ij})$ to obtain a representation of the group $SO(n)$ in V? Since we are dealing with a finite-dimensional representation we know that we can do the exponentiation to obtain a representation of the covering group $Spin(n)$. In fact, in the present case we have a spin representation [double valued representation of $SO(n)$]. To see this we compute

$$\begin{aligned} e^{\frac{\alpha}{2}\gamma(e_1)\gamma(e_2)} &= \sum \frac{1}{n!}(\alpha/2)^n(\gamma(e_1)\gamma(e_2))^n \\ &= \cos(\alpha/2) + \gamma(e_1)\gamma(e_2)\sin(\alpha/2) \end{aligned}$$

which shows that $T(e^{2\pi\cdot\frac{1}{2}\gamma(e_1)\gamma(e_2)}) = -1$ whereas $e^{2\pi s_{12}} = +1$.

The representation T of $Spin(n)$ in V is reducible. The operator γ_{2m+1} commutes with all the generators $T(s_{ij})$ and therefore the eigenspaces of the chirality operator are invariant under $Spin(n)$. The square of γ_{2m+1} is one and so the eigenvalues are ± 1. It is easy to construct the corresponding eigenspaces $V_\pm$. Since γ_{2m+1} anticommutes with each γ_k, $1 \le k \le n$, it anticommutes also with the creation operators a_k^*. The vacuum is an eigenvector of γ_{2m+1} corresponding to the eigenvalue $+1$ and consequently the eigenspace V_+ consists of vectors obtained by acting by a polynomial of even degree in the creation operators to the vacuum whereas V_- is generated by polynomials of odd degree. Both subspaces are of dimension 2^{m-1}. For example, if $n = 4$ then the represention of $Spin(4) = SU(2) \times SU(2)$ in V splits into a pair of two-dimensional representations; these representations are just the defining representations of the two $SU(2)$ subgroups.

In the odd dimensional case we can extend the representation of $Spin(2m)$ in V to a representation of $Spin(2m+1)$ in the same vector space by using the chirality operator. The missing elements of the Lie algebra of $Spin(2m+1)$ are

$$T(s_{i,2m+1}) = \frac{1}{2}\gamma_i\gamma_{2m+1}, \ 1 \le i \le 2m.$$

Exercise 6.1.8. Show that the representation of $Spin(2m+1)$ above is irreducible.

6.2. The Dirac operator

Let (M, g) be an oriented Riemannian manifold of dimension n. Let FM be the bundle of oriented orthonormal frames in the tangent bundle TM. We shall assume that it has *a spin structure*; that means there is a principal $Spin(n)$ bundle P over M and a covering map

$$\phi : P \to FM, \ \phi(pg) = \phi(p)\pi(g)$$

where $\pi : Spin(n) \to SO(n)$ is the double covering homomorphism. If the frame bundle is trivial we can always choose the trivial spin structure $M \times Spin(n)$ with the obvious covering map. In general a manifold does not need to have a spin structure. A simple example of such a manifold is the complex projective space $\mathbb{C}P^2$ of complex dimension 2; we shall return to this in Section 6.4. A manifold can have several inequivalent spin structures; see Gilkey [1984] for more details.

Let V be the complex vector space carrying the irreducible representation of the Clifford algebra in dimension n. As we saw, there is a spin representation ρ of $Spin(n)$ in V which gives an operator realization for the automorphisms of the Clifford algebra. Let $E = P \times_\rho V$ be the associated vector bundle over M; it is called *a spin bundle* of M.

Let Γ be a metric compatible connection of FM, that is, in local coordinates we have

$$\nabla_\mu g_{\nu\lambda} = \partial_\mu g_{\nu\lambda} + \Gamma^\eta_{\mu\nu} g_{\eta\lambda} + \Gamma^\eta_{\mu\lambda} g_{\mu\eta} = 0. \tag{6.2.1}$$

Let $\{h^{(1)}, \ldots, h^{(n)}\}$ be a local orthonormal frame in FM. The metric compatibility is equivalent to the condition that the connection expressed in the basis h takes values in the Lie algebra of the orthogonal group, i.e.,

$$\nabla_\mu h^i = \Gamma^j_{\mu i} h^j \text{ with } \Gamma^j_{\mu i} = -\Gamma^i_{\mu j}. \tag{6.2.2}$$

We can now define a connection in the bundle P (and thus canonically also in the associated bundle E) as follows. Let $\hat{h}$ be a local section of P which covers the section h. Since $Spin(n)$ is the double covering of $SO(n)$ their Lie algebras are isomorphic. With respect to the local section $\hat{h}$ the connection is represented by the Lie algebra valued 1-form $\Gamma^j_{\mu i}$. There is a second local section $\hat{h}z$ which covers h, where z belongs to the center of $Spin(n)$. But a gauge transformation of the Lie algebra valued connection form by an element of the center is an identity transformation and we conclude that the connection is well-defined.

Next we define a first order partial differential operator D, *the Dirac operator,* acting on sections of E. Locally, a section of E is a pair $\psi = (\hat{h}, \xi)$, where $\hat{h}$ is a local section of P and ξ is a locally defined function on M with values in V. We set

$$D\psi = (\hat{h}, \gamma_i h^{(i)\mu} \nabla_\mu \xi), \tag{6.2.3}$$

where $\gamma_i = \gamma(e_i)$ and the $h^{(i)\mu}$'s are the coordinates of the frame h in FM (corresponding to $\hat{h}$ via the canonical projection). To simplify the notation we shall write

$$D\psi = \gamma^\mu \nabla_\mu \psi \tag{6.2.4}$$

with $\gamma^\mu = \gamma_i h^{(i)\mu}$, understanding that a choice of $\hat{h}$ has been made.

Exercise 6.2.5. Show that D indeed is a well-defined operator: If $\hat{h}' = \hat{h}g$ and $\xi' = g^{-1}\xi$ then $D(\hat{h}', \xi')$ as defined by (6.2.3) is the same section as $D(\hat{h}, \xi)$.

Example 6.2.6. Let $M = S^1 \times S^1$ with the flat metric. The frame bundle $FM = M \times SO(2)$ is trivial and we can define a spin structure as $\phi : M \times Spin(2) \to$

$M \times SO(2)$, where $Spin(2) = SO(2)$ and $\phi(x,g) = (x, g^2)$. The spinor representation of the Clifford algebra $C(\mathbb{R}^2)$ is two-dimensional; we can define

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and a complete basis of the Clifford algebra is $\{1, \gamma_1, \gamma_2, \gamma_1\gamma_2\}$. The Dirac operator is now

$$D = \gamma_1 \partial_1 + \gamma_2 \partial_2.$$

A complete set of solutions of the *Dirac equation* $(D + im)\psi = 0$ consists of the $\mathbb{C}^2$ valued functions

$$\psi_p = e^{ip_1\theta_1 + ip_2\theta_2} \begin{pmatrix} 1 \\ \frac{-m}{p_1 + ip_2} \end{pmatrix}, \ \text{with}\, p_1^2 + p_2^2 = m^2$$

where $p_1, p_2 \in \mathbb{Z}$. Thus the eigenvalues of the Dirac operator (the mass of the Dirac particle) are $\pm i\sqrt{p_1^2 + p_2^2}$ and each eigenvalue has a finite multiplicity; the latter is a general property of a Dirac operator on any compact manifold.

Example 6.2.7. Let $M = (S^1)^4$ again with the flat metric. A representation of the Clifford algebra $C(\mathbb{R}^4)$ in $\mathbb{C}^4$ is generated by the matrices

$$\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \ \text{with}\, \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \text{and}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The solutions of the Dirac equation are of the form

$$\psi_p(+) = e^{ip\cdot\theta} \begin{pmatrix} 1 \\ 0 \\ \frac{p_4 - p_3}{m} \\ \frac{ip_2 - p_1}{m} \end{pmatrix}, \ \psi_p(-) = e^{ip\cdot\theta} \begin{pmatrix} 0 \\ 1 \\ \frac{p_1 + ip_2}{-m} \\ \frac{p_3 - p_4}{m} \end{pmatrix}$$

where $m^2 = p \cdot p = p_1^2 + \dots p_4^2$ and $p_i \in \mathbb{Z}$.

Example 6.2.8. Let $M = S^3$. We shall not use the Levi-Cività connection induced by the standard Riemannian metric on the unit sphere but the *flat connection* which can be constructed by observing that S^3 is as a manifold the group $SU(2)$. Now the matrices $\frac{i}{2}\sigma_k$ form an orthonormal basis of the Lie algebra $\mathbf{su}(2)$; a global section of the frame bundle FM is obtained by left translating the tangent vectors $\frac{i}{2}\sigma_k$ at the identity to a general position on the group manifold; we denote the left invariant vector fields by J_k. This defines a metric on S^3. A metric compatible connection is defined by declaring that the left invariant vector fields are parallel, i.e., $\nabla_i \sigma_j = 0$. The Dirac operator can be written as

$$D = \sigma_i J_i.$$

Let $D^j_{m_1 m_2}(g)$ be the matrix elements of $SU(2)$ in an irreducible representation characterized by the spin $j \in \frac{1}{2}\mathbb{N}$. We have then

$$\begin{aligned} iJ_3 D^j_{m_1 m_2} &= m_1 D^j_{m_1 m_2}, \\ iJ_\pm D^j_{m_1 m_2} &= \sqrt{(j \mp m_1)(j \pm m_1 + 1)} D^j_{m_1 \pm 1, m_2}, \end{aligned}$$

where $J_\pm = J_1 \pm iJ_2$. Denoting $\sigma_\pm = \sigma_1 \pm i\sigma_2$ we can write $2D = 2\sigma_3 J_3 + \sigma_+ J_- + \sigma_- J_+$ and we can check that the function

$$\psi_\pm^{jm_1m_2} = \begin{pmatrix} D^j_{m_1m_2} \\ \alpha_\pm D^j_{m_1-1,m_2} \end{pmatrix}, \; j > 0$$

is an eigenvector of D with eigenvalue

$$\lambda_+ = ij, \; \lambda_- = -i(j+1)$$

when $\alpha_\pm = [(\frac{1}{2} - m_1) \pm (j + \frac{1}{2})] \cdot [(j+m_1)(j-m_1+1)]^{-\frac{1}{2}}$. Note that the set of eigenvalues consists of all points $\frac{i}{2}\mathbb{Z}$ *except the point* $-\frac{i}{2}$. The multiplicity of the eigenvalue $\lambda_\pm(j)$ is $(2j+1)^2$ for $j > 0$ but the multiplicity of $\lambda = 0$ is 2 (the eigenvectors being the constant spinors on S^3).

We shall need later the following generalization of the Dirac operator. Let E be a spin bundle over M and let F be some other vector bundle over M with structure group G and connection ω. The tensor product bundle $E \otimes F$ has a connection which is given in terms of a local trivialization through the covariant differentiation $\psi \mapsto (\nabla_\mu + A_\mu)\psi$, where ∇_μ corresponds to the spin connection and A is the $\mathbf{g}$ valued local one-form on M defining the connection ω. A_μ acts only on the second factor of the section $\psi = \psi_E \otimes \psi_F$ of $E \otimes F$. The Dirac operator in the present setting is defined as

$$D = \gamma_i h^{(i)\mu}(\nabla_\mu + A_\mu). \tag{6.2.9}$$

Exercise 6.2.10. Check that (6.2.9) indeed gives a well-defined operator in the space of sections of $E \otimes F$, i.e., $\psi \mapsto D\psi$ does not depend on the gauge choice.

6.3. The determinant of a Dirac operator

The massive Dirac operator

In this subsection we want to clarify what is meant by the determinant of the infinite-dimensional operator $iD + m_0$ in a space $\Gamma(E \otimes F)$ of sections of an extended spin bundle. Here m_0 is a real constant, the mass of the Dirac particle. The reason why the determinant is important in physical applications is that it can be interpreted as *the effective action* for a coupled Dirac-Yang-Mills system. Let us first consider a simple finite-dimensional example. Let T be a positive linear operator in the Euclidean space $\mathbb{R}^n$. Let us compute the integral

$$I(T) = \int e^{-(x,Tx)} d^n x.$$

After the diagonalization $T = S\mathrm{diag}(\lambda_1, \ldots, \lambda_n)S^{-1}$ and the change of variables $y = S^{-1}x$ the integral becomes a product of the Gaussian integrals $\int \exp(-\lambda_i y_i^2) dy_i$ and thus we get

$$I(T) = \prod \sqrt{\frac{\pi}{\lambda_i}} = (\pi)^{n/2} \cdot (\det T)^{-1/2}. \tag{6.3.1}$$

This computation applies only when T is positive. However, we want to apply an infinite-dimensional generalization of (5.3.1) to the Dirac operator which has both positive and negative eigenvalues. We would like to make sense of the integral

$$I = \int e^{-\int \psi^*(iD+m_0)\psi d^n x} d\psi. \tag{6.3.2}$$

When the Dirac operator D is parametrized by a vector potential A this is the effective action $I(A)$ describing a Dirac particle in an external gauge field. In quantum theory the Dirac field ψ is supposed to describe fermions. This means that the components of the spinor ψ should *anticommute* among themselves instead of commuting like the components of ordinary vectors. Thus the correct finite-dimensional analog of (6.3.2) is not the integral (6.3.1) but something like

$$\int_{\psi^*,\psi\in\mathbb{C}^n} e^{-\psi^* T\psi} \tag{6.3.3}$$

where ψ_i^*, ψ_i are elements of a Grassmann algebra. The top element (element of highest degree) in the Grassmann algebra is $\psi_1^* \dots \psi_n^* \psi_1 \dots \psi_n$. We shall *define* the integral (6.3.3) as the coefficient of the top term in the expansion of the exponential; this is

$$\begin{aligned} \frac{(-1)^n}{n!} \sum (\psi_{i_1}^* T_{i_1 j_1} \psi_{j_1}) \dots (\psi_{i_n}^* T_{i_n j_n} \psi_{j_n}) \\ &= -\sum \psi_1^* \dots \psi_n^* T_{1j_1} \dots T_{nj_n} \psi_{j_1} \dots \psi_{j_n} \\ &= -\det T\, \psi_1^* \dots \psi_n^* \psi_1 \dots \psi_n. \end{aligned}$$

Motivated by the finite-dimensional example we define the effective action $I(A)$ to be the determinant of the Dirac operator. However, we still have to define what we mean by $\det(iD + m_0)$. The determinant of a linear operator T in a Hilbert space is well-defined if $T = 1 + S$ where S is a trace-class operator, i.e., the sum of the eigenvalues λ_i of S form an absolutely convergent series. In that case the determinant of T is $\prod(1 + \lambda_i)$ and this converges. In the case of a Dirac operator the eigenvalues do not even form a bounded sequence.

We must introduce a *regularized determinant*. There is a variety of ways to regularize the infinite product of the eigenvalues. We would like to have some sort of continuity of the determinant: The determinant should be expressible as a function $f(\lambda)$ of the set of eigenvalues $\lambda = \{\lambda_k\}$ of $iD+m_0$ such that f is continuous in each argument λ_i (of course f must be such that its value does not depend on the order of the arguments). In addition, we require that f is zero if and only if one of the eigenvalues λ_i vanishes. One simple choice for f is the "cutoff regularization" of the determinant,

$$\det{}^{(M)}(iD + m_0) = \prod_{|\lambda_i|<M} \lambda_i,$$

where $M > 0$ is a "large" cutoff parameter. The obvious disadvantage of this determinant is that it gives no information about the large part of the spectrum. One can introduce the cutoff in a smoother way by choosing a function h on the reals such that $h(x) = x$ for $|x| < M$ and $h(x) \to 1$ very fast as $|x| \to \infty$. A regularized determinant can then be defined as

$$\det{}^h(iD + m_0) = \prod h(\lambda_i).$$

For an elliptic differential operator on a compact manifold such that the real parts of the eigenvalues λ_i are bounded from below (e.g., the Laplace operator) there is a slightly more sophisticated (and more symmetric) way to define the determinant, by the so-called ζ function regularization. Consider the function

$$\zeta(s,\lambda) = \sum_{Re\lambda_i > 0} \lambda^{-s}$$

One can show that this function is holomorphic in the half-plane $\mathrm{Re}\, s > s_0$ for large enough s_0. Furthermore, it extends holomorphically to a regular function at the point $s = 0$. Let $\lambda_1, \ldots, \lambda_p$ be the set of eigenvalues with $\mathrm{Re}\,\lambda_i \leq 0$. The ζ-regularized determinant is then defined as

$$\lambda_1 \lambda_2 \ldots \lambda_p \exp\left[-\frac{d}{ds}\zeta(s,\lambda)\right]|_{s=0} \tag{6.3.4}$$

It is a simple exercise to show that in the finite-dimensional case (6.3.4) gives the usual determinant.

The chiral case

Let the dimension n of M be even, $n = 2m$. The massless Dirac operator ($m_0 = 0$) D anticommutes with the chirality operator γ_{2m+1}. Let S_+ (respectively, S_-) be the subspace of positive (respectively, negative) chirality spinor fields; *a chiral field*, or a *Weyl spinor field*, is a Dirac spinor field which takes values only in one of the two different eigenspaces of the chirality operator. By the remark above, the Dirac operator maps S_+ to S_- and S_- to S_+. We can thus write $D = D_+ + D_-$, with $D_\pm : S_\pm \to S_\mp$. $D_\pm$ are the *Weyl operators* on M. In the massless case we can study the coupling of a vector potential A independently to either of the chiral fields and therefore it would be natural to define the determinant (effective action) for the Weyl operators. However, besides the regularization, we run immediately into a problem: The Weyl operators are operators between *different spaces* $S_\pm$ and therefore it does not make sense to speak about the eigenvalue problem for $D_\pm$.

There is a way out of this dilemma. Instead of trying to define the determinant of D_+ directly we define it relative to a fixed operator $T : S_- \to S_+$. For example, if the vector bundle F is trivial (we keep the notation of Section 5.2) we could take $T = D_-^0$ as the free Weyl operator determined by the flat connection $A = 0$ in F. Since T does not depend on the vector potential A the determinant $\det TD_+(A)$, considered as a function of A, is a good replacement for the ill-defined determinant $\det D_+(A)$. Of course, in order that the determinant does not vanish identically, we must choose T such that it has no zero modes.

Example 6.3.5. Let $M = S^1 \times S^1$ with the flat metric and let $F = M \times \mathbb{C}^N$. As the gauge group we take $G = U(N)$ with the obvious action in F. The fibers of $S_\pm$ can now be identified as $\mathbb{C} \otimes \mathbb{C}^N = \mathbb{C}^N$. The free Weyl operator is now

$$D_-^0 = \frac{\partial}{\partial\theta_1} + i\frac{\partial}{\partial\theta_2}$$

and its only zero modes on the torus are the constant functions. Adding to D_-^0 a small "mass" $0 < \epsilon$, $T = D_-^0 + \epsilon$, we obtain an operator $T : S_- \to S_+$ without zero modes. We have

$$TD_+(A) = \partial_1^2 + \partial_2^2 + \text{lower order terms}.$$

The spectrum of this elliptic operator behaves essentially like the spectrum of the Laplace operator (for large eigenvalues). We can apply for example the ζ function regularization to define the determinant of $TD_+(A)$. The actual computation of the determinant is a complicated matter and no simple closed form of it is known.

Remark. If it happens that the Weyl operator D_-^0 does not have zero modes then one can use instead of $D_-^0 D_+$ the operator $(D_-^0)^{-1}D_+ = 1 + (D_-^0)^{-1}\gamma^\mu A_\mu$. The advantage with this operator is that it has a generalized determinant $\det_p$ for high enough $p > 1$; we shall discuss these determinants in detail in the next chapter.

For a fixed metric on M but for a variable connection ω in F the regularized determinant $\det_{reg}$ of the Dirac operator is a complex valued function in the space $\mathcal{A}$ of $\mathbf{g}$ valued connections in F. Next we want to discuss the *gauge dependence* of this function. In order to avoid some unessential technical complications we shall assume that the bundle F is trivial. Now a connection in F is a globally defined $\mathbf{g}$ valued 1-form A on M. A gauge transformation is a mapping of A onto itself, $A \mapsto A^g = gAg^{-1} + dgg^{-1}$, where $g : M \to G$ is a smooth function. Under a gauge transformation g a Dirac spinor field ψ is mapped to the field $[T(g)\psi](x) = g(x)\psi(x)$. Since the Dirac operator is constructed in terms of covariant derivatives we have

$$T(g)D(A)T(g^{-1}) = D(A^g)$$

and therefore, if ψ is an eigenvector of $D(A)$ corresponding to the eigenvalue λ, then $T(g)\psi$ is an eigenvector of $D(A^g)$ corresponding to the same eigenvalue. The spectra of $D(A)$ and $D(A^g)$ are identical and consequently the regularized determinants of $D(A)$ and $D(A^g)$ are the same. The Dirac determinant is thus really a function in the quotient $\mathcal{A}/\mathcal{G}$, the space of vector potentials modulo gauge transformations.

The situation is different with the Weyl operator $D_+(A)$ [or $D_-(A)$]. The Weyl operator $D_+(A)$ between S_+ and S_- transforms equivariantly with respect to the gauge transformations but we have to consider the operator $TD_+(A)$ and this does not transform equivariantly in general. Thus there is no reason why the "determinant" of the Weyl operator should be gauge invariant. In physics literature the variation of the determinant under the group of gauge transformations is called the *chiral anomaly.* The manner in which the determinant transforms under the gauge group depends on the regularization chosen. There is an important invariant which does not depend on the regularization, namely, *the cohomology class* of the chiral anomaly: Let $f(A)$ be the Weyl determinant, defined by some regularization. Then

$$c(A;g) = \frac{f(A^g)}{f(A)}$$

is a function on $\mathcal{A} \times \mathcal{G}$ taking values in the multiplicative group $\mathbb{C}^\times$. The quotient is well-defined since $D_+(A^g)$ has precisely the same zero-modes as the operator $D_+(A)$. By definition, c is a one-cocycle of $\mathcal{G}$ with values in the group of $\mathbb{C}^\times$ valued functions on $\mathcal{A}$,

$$c(A^g;g')c(A;g) = c(A;gg').$$

It is the class $[c] \in H^1(\mathcal{G})$ which is regularization independent. We shall sketch the proof below.

The Dirac determinant bundle

A cocycle c defines a line bundle DET over $\mathcal{A}/\mathcal{G}$, the *determinant bundle*; the sections of the dual determinant bundle DET^* are complex valued functions ψ on $\mathcal{A}$ such that

$$\psi(A^g) = \psi(A)\, c(A; g). \tag{6.3.6}$$

The equivalence classes of line bundles correspond to cohomology classes of cocycles c. In order to classify these line bundles we shall study the topology of the base space $\mathcal{A}/\mathcal{G}$. The topology depends on the topology of the manifold M but we shall consider here only the simple case when $M = S^n$. Choose a point $N \in S^n$ and a disk $D' \subset S^n$ with N as the center. For any $A \in \mathcal{A}$ there is a smooth $f_A : S^n \to G$ such that $f(N) = 1$ and such that the radial component of the gauge transformed vector potential A^f vanishes in the disk D'. The function f_A is uniquely defined *inside of the disk* D'. The complement of D' in S^n is also a disk, to be denoted by D. Denote by h_A the restriction of f_A to the disk D. Now h_A is not uniquely defined; it depends on the choice of f_A. However, the class $[h_A]$, h_A modulo the group $\mathcal{G}_D$ of gauge transformations $g : D \to G$, with $g = 1$ on the boundary of D, is uniquely defined. We have thus a mapping $\mathcal{A} \to \mathcal{A}_D/\mathcal{G}_D$, $A \mapsto q(A) = A^{h_A}|_D \mod \mathcal{G}_D$.

Let $g \in \mathcal{G}$ and $A \in \mathcal{A}$. Now $f_{A^g} = f_A g^{-1}$ but $q(A^g) = q(A)$ and therefore the map $q : \mathcal{A} \to \mathcal{A}_D/\mathcal{G}_D$ gives also a well-defined map $q : \mathcal{A}/\mathcal{G} \to \mathcal{A}_D/\mathcal{G}_D$. This map is a *homotopy equivalence.* This means that there is a map p in the opposite direction such that qp and pq can be continuously deformed to the identity maps. The map p is constructed as follows. Let B be a vector potential in the disk D. Extend this to a vector potential A in S^n by defining a potential B' in D' as $B'(r, \theta) = rB(1, \theta)$, where θ denotes collectively the angular coordinates in the disk D' and the boundary values of B' are defined to be equal to the boundary values of B on the common boundary. Now A is defined by patching together B and B'. Suppose that $C = B^h$ for some $h \in \mathcal{G}_D$. The boundary values of C and B are equal and therefore the vector potential $p(B) = A$ is obtained from $p(C)$ by a gauge transformation g which is equal to 1 in D' and coincides with h in D'. It follows that p gives really a map from $\mathcal{A}_D/\mathcal{G}_D$ to $\mathcal{A}/\mathcal{G}$. We shall show that pq is homotopic to the identity map on $\mathcal{A}/\mathcal{G}$. Consider the following one-parameter family of mappings on $\mathcal{A}/\mathcal{G}$:

$$\rho_t(A) = \begin{cases} A^{f_A}|_D \text{ on } D \\ tA^{f_A}|_{D'} + (1-t) r A^{f_A}|_{\partial D} \text{ on } D' \end{cases}.$$

We have $\rho_1 = id$ and $\rho_0 = pq$. This is the required homotopy.

Exercise 6.3.7. Show that qp is homotopic to the identity mapping on $\mathcal{A}_D/\mathcal{G}_D$.

Next we shall show that $\mathcal{A}_D/\mathcal{G}_D$ is homotopic to $S^{n-1}G$. (In general $S^m G$ is disconnected; we shall reserve here this notation for the connected component of identity.) For any vector potential A in the disk D we choose a mapping $u_A : D \to G$ such that A^{u_A} is in the radial gauge (the radial component of the transformed potential vanishes) and that $u_A(0) = 1$, where the center of D is denoted by 0. The function u_A is uniquely defined. A gauge transformation of A by an element g of $\mathcal{G}_D$ multiplies the map u_A by g^{-1} from the right and it follows that $A \mapsto u_A|_{\partial D}$ defines a map $i : \mathcal{A}_D/\mathcal{G}_D \to S^{n-1}G$. This is a homotopy equivalence. The homotopy inverse j is obtained as

$$j(f) = d\hat{f}\hat{f}^{-1}, \text{ with } \hat{f} \text{ any extension of } f \text{ to } D.$$

The family $\rho_t(A) = tA + (1-t)du_A u_A^{-1}$, $0 \le t \le 1$, of mappings of $\mathcal{A}_D/\mathcal{G}_D$ onto itself is a homotopy connecting ji to the identity mapping.

Exercise 6.3.8. Show that ij is homotopic to the identity on $S^{n-1}G$.

Combining the homotopy equivalences q and i we obtain a homotopy equivalence $\mathcal{A}/\mathcal{G} \simeq S^{n-1}G$. It follows that line bundles over $\mathcal{A}/\mathcal{G}$ can be classified in terms of line bundles over $S^{n-1}G$; the (equivalence classes of) line bundles over the former space are obtained as pull-backs of line bundles over the latter space. The problem reduces then to a study of the second de Rham cohomology of $S^{n-1}G$. The cohomology depends both on n and the group G. If $n=2$ the first homotopy group of $S^{n-1}G$ is isomorphic with the second homotopy group of G which is always zero. By the Hurewicz theorem the second cohomology of $S^{n-1}G$ is now isomorphic with the second homotopy which in turn is equal to the third homotopy of G. For any compact simple Lie group $\pi_3 G = \mathbb{Z}$. The classification of the determinant bundles in the case of $M = S^2$ corresponds to the classification of the central extensions of the loop group S^1G. If $n=4$ then the first homotopy group of $S^{n-1}G$ is equal to the fourth homotopy of G which is zero if G is a simple compact group of rank bigger than one. In that case again the second cohomology of $S^{n-1}G$ is equal to the second homotopy which is equal to the fifth homotopy group of G which is equal to $\mathbb{Z}$ for $G = SU(N), N > 2$.

We have considered here only the case when the bundle F is trivial. However, it is not difficult to show that in the general case $\mathcal{A}/\mathcal{G}$ is homotopic to an appropriate component of the disconnected group of *all* smooth maps $S^3 \to G$.

In fact, we already have an explicit construction of the nontrivial line bundles over S^3G in Section 4.3. Over the identity component S^3_0G of S^3G the sections of the line bundle are by definition complex valued functions ψ on D_4G which satisfy the condition

$$\psi(fg) = \psi(f) \cdot e^{2\pi i \omega(f;g)}$$

where ω is a 1-cocycle for the natural right action of S^4G on D_4G (we have again identified S^4G as the subgroup of D_4G consisting of functions g which are equal to 1 on the boundary). The cocycle is given by

$$\omega(f;g) = \int c^{1,5}(f^{-1}df; g),$$

where the integral is computed over a five-dimensional disk D_5 with boundary S^4. The sections of the pull-back bundle DET^* over $\mathcal{A}/\mathcal{G}$ are functions on $\mathcal{A}$ which satisfy the cocycle condition

$$\psi(A^g) = \psi(A) \cdot \exp\left[2\pi i \int c^{1,5}(A;g)\right], \; g \in \mathcal{G} = S^4G.$$

The first Chern class c_1 of the line bundle DET^* is the generator of the cohomology group $H^2(\mathcal{A}/\mathcal{G};\mathbb{Z})$. (Elements of the second cohomology group with integral coefficients can be thought of here as equivalence classes of closed two-forms such that the integral of the form over any compact surface without boundary is 2π times an integer.) The line bundle with Chern class equal to n times this basic Chern class is obtained simply by replacing $c^{1,5}$ by $nc^{1,5}$.

To show that the determinant bundle of the Weyl operator really is twisted one must also do some analysis. But the only thing necessary is to show that the phase of the regularized determinant $\det_{reg}(D_0^* D_A)$ winds around zero for a suitable loop of gauge transformations $A \mapsto A^{g(t)}$, $0 \le t \le 2\pi$; let us assume that this is the case. (This follows from an explicit evaluation of the variation of the determinant; see Atiyah and Singer [1984].) Now if there is a function $h : \mathcal{A} \to \mathbb{C}^\times$ such that the gauge variation of the regularized determinant can be written as $h(A^g)/h(A)$ then the phase could not wind around zero along any loop in the space of gauge transformations since the space $\mathcal{A}$ is contractible. This shows that the cocycle $\exp[2\pi i \omega(A; g)]$ is nontrivial. To show that the deteminant bundle corresponds to a value n of the Chern class one has to show that the determinant of the Weyl operator winds exactly n times around zero along a noncontractible loop in $\mathcal{G}$ which generates $\pi_1(\mathcal{G})$.

6.4. On the geometry of the Dirac determinant bundle

Curvature and anomalies

There is a close connection between the chiral anomaly and the curvature of the moduli space $\mathcal{A}/\mathcal{G}$. We shall explain the main results without proofs.

Let M be a compact connected oriented spin manifold with a fixed Riemannian metric g. We assume that the dimension of M is even, say $2n$. Let P be a principal G bundle over M with G a compact Lie group. The group $\mathcal{G}$ of gauge transformations consists by definition of all automorphisms of the bundle P which descend to an identity transformation on the base M. If P is trivial, $P = M \times G$, then the elements of $\mathcal{G}$ are just G valued functions f on M; the action of f in P is then given by $(x, g) \mapsto (x, f(x)g)$. In order that the moduli space $\mathcal{A}/\mathcal{G}$ is a smooth manifold we shall fix a point $p_0 \in P$ and consider only those gauge transformations which leave p_0 fixed. The group $\mathcal{G}$ acts in the space $\mathcal{A}$ of connections in P in the natural way, i.e., through pull-back of the one forms $A \in \mathcal{A}$. We have then an action of $\mathcal{G}$ in $P \times \mathcal{A}$ which commutes with the action of the structure group G in P. We have now a principal $\mathcal{G}$ bundle $P \times \mathcal{A}$ over $\mathcal{L} = (P \times \mathcal{A})/\mathcal{G}$. We can further divide by G (since G commutes with $\mathcal{G}$) and we obtain a principal G bundle $\mathcal{L}$ over $M \times \mathcal{A}/\mathcal{G}$.

Let us define a connection ω in the bundle $\mathcal{L}$. Note first that the space $P \times \mathcal{A}$ has a natural metric: If (p, A) is a point in $P \times \mathcal{A}$ and (t_i, B_i) $(i = 1, 2)$ is a pair of tangent vectors at (p, A) then the inner product is

(6.4.1)
$$\begin{aligned} &< (t_1, B_1), (t_2, B_2) > \\ &\qquad = g(h(t_1), h(t_2)) + (v(t_1), v(t_2)) + \int_M (B_1^\mu, B_{2\,\mu}) d(vol_M), \end{aligned}$$

where we have used a fixed invariant inner product $(\cdot, \cdot)$ in the Lie algebra of G and h, v are the horizontal and vertical projections defined by the connection A. The Riemannian metric is written as $g(t_1, t_2) = g^{\mu\nu} t_{1\,\mu} t_{2\,\nu} = t_1^\mu t_{2\,\mu}$. The integration can be actually carried out in the base manifold M since the variations B_i of a connection transform homogeneously in gauge transformations, $B_i \mapsto f B_i f^{-1}$. The vertical vectors are identified as elements of $\mathbf{g}$ in the usual way. The metric (6.4.1) descends to a G invariant metric on $\mathcal{L}$: With respect to a local trivialization the sum of the first two terms on the right-hand-side of (6.4.1) is

$$g(u_1, u_2) + (X_1 - u_{1\mu} A_\mu, X_2 - u_{2\mu} A_\mu)$$

where $u_1, u_2 \in T_{\pi(p)}M$, X_1, X_2 are in $\mathbf{g}$ and (A_μ) is a local one-form on M describing the connection A. In a gauge transformation f the point x is left invariant, X transforms as $X \mapsto f(x)Xf(x)^{-1} + t_\mu A_\mu(x)$, and $A_\mu \mapsto fA_\mu f^{-1} + \partial_\mu f\, f^{-1}$. The difference $X - t \cdot A$ transforms according to the adjoint representation and by the invariance of the inner product in $\mathbf{g}$, (6.4.1) is invariant under $\mathcal{G}$. The invariance under G follows immediately from the equivariantness of the connection forms and the invariance of the inner product in $\mathbf{g}$.

The connection ω in $\mathcal{L}$ is given by declaring that the horizontal subspace at a given point is the orthogonal complement of the vertical subspace with respect to the Riemanniann metric $< \cdot, \cdot >$ on $\mathcal{L}$.

Exercise 6.4.2. Show that the distribution of horizontal subspaces given above really defines a connection.

Let us compute the curvature of the connection ω. Let $(x, [A])$ be a point in $M \times \mathcal{A}/\mathcal{G}$ and let (u_i, B_i) be a pair of tangent vectors at $(x, [A])$. The tangent vectors to $\mathcal{A}/\mathcal{G}$ can be represented by vectors in the space of $\mathbf{g}$ valued one-forms on M by fixing the gauge: For a given representative A of the gauge class $[A]$ a tangent vector is a form B_μ such that $\partial_\mu B^\mu - [A_\mu, B^\mu] = 0$. This is precisely the condition that the form B_μ is orthogonal to the gauge orbit through the point A. Namely, a tangent vector along the gauge orbit is a form $C = [X, A_\mu] + \partial_\mu X$ (where X is a $\mathbf{g}$ valued function on M) and by partial integration

$$
\begin{aligned}
< B, C > &= \int (B^\mu, [X, A_\mu] + \partial_\mu X) d(vol_M) \\
&= \int (X, [A_\mu, B^\mu] - \partial_\mu B^\mu) d(vol_M).
\end{aligned}
$$

This has to be zero for all X which implies the *background gauge* condition above. We can split the curvature Ω as $\Omega^{2,0} + \Omega^{1,1} + \Omega^{0,2}$ corresponding to the splitting of the tangent spaces of $M \times \mathcal{A}/\mathcal{G}$ to the tangent space of M and $\mathcal{A}/\mathcal{G}$. It is not difficult to see that

$$
\Omega^{2,0}(u_1, u_2) = F(u_1, u_2), \qquad \Omega^{1,1}(u, B) = u_\mu B_\mu
$$

where B is a tangent vector at A (in the background gauge) to $\mathcal{A}/\mathcal{G}$ and the u's are tangent vectors at $x \in M$; F is the curvature form corresponding to the vector potential A.

In order to evaluate $\Omega^{0,2}$ we construct the connection form in the $\mathcal{G}$ bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. The vertical subspace at A consists of all one- forms of the type $[X, A_\mu] + \partial_\mu X$, where $X : M \to \mathbf{g}$ is an infinitesimal gauge transformation. The horizontal subspace consists of the vector potentials B in the background gauge with respect to the base point A. The horizontal projection of an arbitrary tangent vector B at A is

$$
h(B_\mu) = B_\mu - D_\mu^A \Delta_A^{-1} D_\nu^A B^\nu
$$

where $D_\mu^A Z \equiv \partial_\mu Z - [A_\mu, Z]$ and $\Delta_A = D_\mu^A D^{A\mu}$ is the covariant Laplacian. It follows that the connection form with respect to a given local trivialization of the bundle $\mathcal{A}$ is given by

$$
B \mapsto \Delta_A^{-1} D_\mu^A B^\mu. \tag{6.4.3}
$$

Evaluating this at the point $x \in M$ we get the value of the connection form in $\mathcal{L}$ in the direction of the tangent vector $(0, B)$. The curvature of this gives

$$\Omega^{0,2}(B, B') = \Delta_A^{-1}[B, B'], \tag{6.4.4}$$

where B, B' are in the background gauge.

The curvature formula at hand we can compute the Chern classis of the vector bundle $\mathcal{E} = \mathcal{L} \times_\rho \mathbb{C}^N$ over $M \times \mathcal{A}/\mathcal{G}$, where ρ is a representation of G in $\mathbb{C}^N$. Let us assume for simplicity that $M = S^{2n}$. We shall use some results from Atiyah-Singer index theory. Let $D(A)_\pm$ be the Weyl operators on M defined by a vector potential A. Consider the kernels $\ker D(A)_\pm$. For a given A these are finite-dimensional vector spaces. However, the family of vector spaces $\ker D(A)_\pm$ do not form a vector bundle over $\mathcal{A}/\mathcal{G}$ since the dimension of the kernels may jump. Instead, their formal difference $K = \ker D_+ - \ker D_-$ is defined in the sense of K theory; the difference is not a vector bundle in the ordinary sense but it makes sense to speak about characteristic classes of K. It follows from the families index theorem in Atiyah and Singer [1971] that the Chern classes of K can be evaluated by integrating the Chern character of $\mathcal{E}$ over the first factor M. To be more precise, one has the following theorem [Atiyah and Singer, 1984]:

Theorem 6.4.5. *The Chern character of K is*

$$ch(K) = \int_M ch(\mathcal{E}).$$

If M is not a sphere then there is a correction to the above formula from a characteristic class associated to the spin bundle of M. In order to illustrate the use of this result we give more explicit formulas in two particular cases. Let $M = S^4$ and $G = SU(N)$ (so that $\mathrm{tr} F = 0$.) The 0:th Chern class of K is simply

$$\frac{-1}{8\pi^2} \int_M \mathrm{tr}\, F \wedge F$$

which is thus the Chern number computed from the second Chern class of the bundle $P \times_G \mathbb{C}^N$. The first Chern class of K is the 2-form

$$\begin{aligned} c_1(B, B') =& \frac{-i}{24\pi^3} \int_M \mathrm{tr}\, \Omega^{4,2}(B, B') \\ =& \frac{-i}{24\pi^3} \int_M \epsilon^{\alpha\beta\gamma\delta} \mathrm{tr} \left\{ F_{\alpha\beta} F_{\gamma\delta} \Delta_A^{-1}[B^\mu, B'_\mu] \right. \\ & \left. + F_{\alpha\beta} \Delta_A^{-1}[B^\mu, B'_\mu] F_{\gamma\delta} + F_{\alpha\beta}(B_\gamma B'_\delta + B'_\gamma B_\delta) \right\}. \end{aligned}$$

To any multiple $nc_1(K)$ of the first Chern class one can associate a complex line bundle over the parameter space $\mathcal{A}/\mathcal{G}$. The curvature of the line bundle is the differential form $nc_1(K)$. Because of $H^2(\mathcal{A}/\mathcal{G}; \mathbb{Z}) = \mathbb{Z}$ [in the case $M = S^4$ and $G = SU(N)$, $N > 2$] these line bundles must be the same as the bundles described by the cocycles c in (6.3.6), corresponding to the chiral anomaly of the Dirac operator. Thus we have a direct relation between the curvature of $\mathcal{A}/\mathcal{G}$ and the chiral anomaly.