CHAPTER 2: DIFFERENTIAL FORMS

2.1 Multilinear forms

Let V be a vector space, $\dim V = n < \infty$, over the field $K = \mathbb{R}$ or $K = \mathbb{C}$. The *dual space* V^* consists of all linear functions $f : V \to K$ and it is a vector space under the usual addition and scalar multiplication of functions. If $\{e_1, \dots, e_n\}$ is a basis of V then we can define a basis $\{f_1, \dots, f_n\}$ of V^* by $f_i(e_j) = \delta_{ij}$. We denote $\Omega^1(V) = V^*$.

Next we define $\Omega^2(V) = V^* \wedge V^*$ as the space of antisymmetric functions $f : V \times V \to K$ which are linear in both arguments. $\Omega^2(V)$ is a vector space of dimension $n(n-1)/2$. A basis is given by the functions f_{ij} defined by

$$f_{ij}(e_k, e_l) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

with $1 \le i < j \le n$. We set $f_{ji} = -f_{ij}$. A general element of $\Omega^2(V)$ is then a linear combination $f = a_{ij} f_{ij}$ with $a_{ij} = -a_{ji}$, that is, elements in $\Omega^2(V)$ are antisymmetric tensors on V^*.

If $f, g \in \Omega^1(V)$ then $f \wedge g \in \Omega^2(V)$ with $(f \wedge g)(x, y) = f(x)g(y) - f(y)g(x)$. In particular, $f_{ij} = f_i \wedge f_j$. The wedge product is antisymmetric, $f \wedge g = -g \wedge f$.

Example When $V = \mathbb{R}^3$ the wedge product is simply the cross product of vectors. We can identify $\Omega^2(\mathbb{R}^3)$ as the space $\mathbb{R}^3$ by using the standard basis: The elements in an antisymmetric tensor (a_{ij}) are parametrized by a vector (a_{23}, a_{31}, a_{12}) and then $x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$.

In general, $\Omega^k(V)$ denotes the space of alternating multilinear forms $f : V \times V \times \dots \times V \to K$ (k arguments). Alternating means that the sign of the function is reversed when a pair of arguments is transposed. In other words, a permutation σ of the arguments can be compensated by a multiplication by $\epsilon(\sigma)$ where $\epsilon(\sigma) = \pm 1$ is the parity of the permutation,

$$f(x_{\sigma(1)}, ..., x_{\sigma(n)}) = \epsilon(\sigma) f(x_1, ..., x_n).$$

A basis in $\Omega^k(V)$ is given by the multilinear forms $f_{i_1 i_2 \dots i_k}$ defined by

$$f_{i_1 \dots i_k}(x^{(1)}, \dots, x^{(k)}) = \det\,(x^{(j)}_{i_m}).$$

By the symmetry properties of the determinant the right-hand-side indeed defines an alternating form. The dimension of $\Omega^k(V)$ is equal to the binomial factor $\binom{n}{k}$. In particular, dim $\Omega^n(V) = 1$ and $\Omega^k(V) = 0$ for $k > n$. We set $\Omega^0(V) = K$ and

$$\Omega(V) = \Omega^0(V) \oplus \Omega^1(V) \oplus \cdots \oplus \Omega^n(V).$$

The dimension of the direct sum is

$$\dim \Omega(V) = \sum_k \binom{n}{k} = 2^n.$$

We generalize the wedge product to a product

$$\Omega^j(V) \times \Omega^k(V) \to \Omega^{j+k}(V)$$

by the formula

$$(f \wedge g)(x^{(1)}, \ldots, x^{(j+k)}) = \frac{1}{j!}\frac{1}{k!} \sum_{\sigma \in S_{j+k}} \epsilon(\sigma) f(x^{(\sigma(1))}, \ldots, x^{(\sigma(j))}) g(x^{(\sigma(j+1))}, \ldots, x^{(\sigma(j+k))}),$$

where S_n is the group of permutations of integers $1, 2, \ldots, n$.

Exercise 1 Show that $f \wedge g$ is alternating.

Exercise 2 Prove that $f \wedge g = (-1)^{jk} g \wedge f$.

Exercise 3 Prove that $f \wedge (g \wedge h) = (f \wedge g) \wedge h$.

Note that the basis $f_{i_1 i_2 \ldots i_k}$ defined above is obtained from the f_i's,

$$f_{i_1 i_2 \ldots i_k} = f_{i_1} \wedge f_{i_2} \wedge \cdots \wedge f_{i_k}.$$

2.2 Differential forms

Let M be a smooth manifold of dimension n. A *differential form* of degree k on M is a smooth distribution $\omega_x \in \Omega^k(T_x M)$ of alternating forms in the tangent spaces. We denote by $\Omega^k(M)$ the set of differential forms of degree k. Smoothness of the distribution $x \mapsto \omega_x$ is defined in terms of local coordinates $x_1, \ldots, x_n$. Recall that each coordinate x_i defines a local vector field $\partial^i = \frac{\partial}{\partial x_i}$, interpreted as a derivation of the algebra $C^\infty(M)$. A tangent vector at a point x is uniquely written as $v = v_i \partial^i$. For this reason ω is given in terms of the coordinate functions

$$\omega^{i_1 \ldots i_k}(x) = \omega_x(\partial^{i_1}, \ldots, \partial^{i_k}).$$

Smoothness of ω means that the coordinate functions $\omega^{i_1\dots i_k}(x)$ are smooth functions of the coordinates x_i.

Locally, a basis for $\Omega^1(M)$ is given by the differential 1-forms dx_i defined by

$$dx_i(\partial^j) = \delta_{ij}.$$

A basis for k-forms is given by

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \text{ with } 1 \le i_1 < i_2 < \cdots < i_k \le n.$$

In the given coordinate chart we have then

$$\omega = \frac{1}{k!}\omega^{i_1 i_2\dots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

The wedge pruduct of forms $\omega \in \Omega^j(M)$ and $\theta \in \Omega^k(M)$ is a form in $\Omega^{j+k}(M)$ defined pointwise as $(\omega \wedge \theta)_x = \omega_x \wedge \theta_x$. The product is associative and

$$\omega \wedge \theta = (-1)^{jk}\theta \wedge \omega.$$

The *exterior derivative* of $\omega \in \Omega^k(M)$ is defined in terms of local coordinates as an element $d\omega$ of $\Omega^{k+1}(M)$,

$$(1) \qquad d\left(\omega^{i_1\dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = \partial^j \omega^{i_1\dots i_k} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

We define $\Omega^0(M) = C^\infty(M)$ and then

$$df = \partial^j f dx_j$$

for a smooth function f. We must also check that the definition of $d\omega$ in terms of local coordinates is compatible with coordinate tranformations. Since $\partial'^k = \frac{\partial}{\partial x'_k} = \frac{\partial x_j}{\partial x'_k}\partial^j$ by the chain rule, we obtain

$$\omega'^{i_1\dots i_k} = \omega(\partial'^{i_1}, \dots, \partial'^{i_k}) = \omega(\partial^{j_1}, \dots, \partial^{j_k})\frac{\partial x_{j_1}}{\partial x'_{i_1}} \cdots \frac{\partial x_{j_k}}{\partial x'_{i_k}}.$$

In other words,

$$\omega^{i_1\dots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega'^{j_1\dots j_k} dx'_{j_1} \wedge \cdots \wedge dx'_{j_k},$$

for

$$\omega'^{i_1\dots i_k} = \omega^{j_1\dots j_k}\frac{\partial x_{j_1}}{\partial x'_{i_1}} \cdots \frac{\partial x_{j_k}}{\partial x'_{i_k}}.$$

When the exterior differentiation is applied to the right-hand-side we obtain an expression similar to (1) but the coordinates x_i replaced by x'_i; But the exterior derivative of the right-hand-side is equal to

$$\partial'^j \omega'^{i_1 \ldots i_k} dx'_j \wedge dx'_{i_1} \wedge \cdots \wedge dx'_{i_k} = \frac{\partial x_l}{\partial x'_j} \partial^l \left(\frac{\partial x_{j_1}}{\partial x'_{i_1}} \cdots \frac{\partial x_{j_k}}{\partial x'_{i_k}} \omega^{j_1 \ldots j_k} \right) dx'_j \wedge dx'_{i_1} \wedge \cdots \wedge dx'_{i_k}$$
$$= \partial^l \omega^{i_1 \ldots i_k} dx_l \wedge dx_{i_1} \cdots \wedge dx_{i_k} + \omega^{j_1 \ldots j_k} \frac{\partial^2 x_{j_1}}{\partial x'_j x'_{i_1}} \frac{\partial x_{j_2}}{\partial x'_{i_2}} \cdots \frac{\partial x_{j_k}}{\partial x'_{i_k}} dx'_j \wedge dx'_{i_1} \cdots \wedge dx'_{i_k} + \ldots.$$

Using the antisymmetry of the wedge products $dx_j \wedge dx_{i_p}$ and the symmetry of the second derivatives we observe that all the terms involving second derivatives are identically zero and therefore only the first term remains, giving the exterior derivative of ω in the x_i coordinates.

To remember the transformation rule for differential forms it is sufficient to keep in mind the transformation for 1-forms,

$$dx'_i = \frac{\partial x'_i}{\partial x_j} dx_j,$$

since the higher order forms are exterior products of the basic 1-forms and smooth functions.

Theorem. $d^2 = 0$.

Proof.

$$d^2(\omega^{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d\left(\partial^j \omega^{i_1 \ldots i_k} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$
$$= \partial^l \partial^j \omega^{i_1 \ldots i_k} dx_l \wedge dx_j \wedge \ldots dx_{i_k}.$$

Again, using the symmetry of second derivatives and antisymmetry of the wedge product $dx_l \wedge dx_j$ we see that all terms on the right vanish and thus $d^2\omega = 0$.

Note that $d\omega = 0$ for $\omega \in \Omega^0(M)$ implies that ω is a constant function in each connected component of M. Set $\Omega(M) = \Omega^0(M) \oplus \Omega^1(M) \oplus \ldots \Omega^n(M)$ with $n = \text{dim}M$.

Theorem. *Let* $\omega \in \Omega^p(M)$ *and* $\theta \in \Omega^q(M)$*. Then* $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta$.

Proof. Set $\omega = \omega^{i_1 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$ and $\phi = \phi^{j_1 \dots j_q} dx_{j_1} \wedge \dots \wedge dx_{j_q}$. Then

$$\begin{aligned}
\omega \wedge \phi &= \omega^{i_1 \dots i_p} \phi^{j_1 \dots j_q} dx_{i_1} \wedge \dots dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q} \\
d(\omega \wedge \phi) &= \phi^{j_1 \dots j_q} \partial^k \omega^{i_1 \dots i_p} dx_k \wedge dx_{i_1} \wedge \dots dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q} \\
&\quad + \omega^{i_1 \dots i_p} \partial^k \phi^{j_1 \dots j_q} dx_k \wedge dx_{i_1} \wedge \dots dx_{j_q} \\
&= d\omega \wedge \phi + (-1)^p \omega^{i_1 \dots i_p} \partial^k \phi^{j_1 \dots j_q} dx_{i_1} \dots dx_{i_p} \wedge dx_k \wedge dx_{j_1} \wedge \dots dx_{j_q} \\
&= d\omega \wedge \phi + (-1)^p \omega \wedge d\phi,
\end{aligned}$$

where we have used the alternating property of the wedge product, $dx_{i_1} \wedge \dots dx_{i_p} \wedge dx_k = (-1)^p dx_k \wedge dx_{i_1} \wedge \dots dx_{i_p}$.

There is alternative way to think about differential forms. Let $X \in D^1(M)$ and $\omega \in \Omega^1(M)$. We can define a smooth function on M by $f(x) = \omega_x(X(x))$ by the natural pairing of tangent vectors $X(x)$ and the elements $\omega_x \in T_x^* M$ in the dual. Thus a 1-form is a map from $D^1(M)$ to $C^\infty(M)$. This map is linear, moreover $\omega(gX) = g\omega(X)$ for any smooth function g.

In a similar way, any $\omega \in \Omega^k(M)$ can be thought of as a multilinear function $\omega : D^1(M) \times D^1(M) \times \dots D^1(M) \to C^\infty(M)$ by

$$\omega(X_1, X_2, \dots, X_k)(x) = \omega_x(X_1(x), \dots, X_k(x)).$$

By the definition of a differential form, this map is alternating.

There is converse result which we state without proof: Any alternating map $D^1(M) \times \dots \times D^1(M) \to C^\infty(M)$ which is $C^\infty(M)$ linear in each variable, is uniquely represented by a differential form.

Let $f \in \Omega^0(M)$ and $X \in D^1(M)$. Then

$$df(X) = (\partial^k f dx_k)(X_j \partial^j) = X_j \partial^k f dx_k(\partial^j) = X_j \partial^j f = X \cdot f.$$

Next let $\omega \in \Omega^1(M)$ and $X, Y \in D^1(M)$. Now

$$\begin{aligned}
(d\omega)(X,Y) &= (\partial^j \omega^i dx_j \wedge dx_i)(X,Y) = (\partial^j \omega^i)(dx_j(X) dx_i(Y) - dx_j(Y) dx_i(X)) \\
&= (\partial^j \omega^i)(X_j Y_i - Y_j X_i) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega^i (X_j \partial^j Y_i - Y_j \partial^j X_i) \\
&= X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y]).
\end{aligned}$$

Exercise Generalize the formula above to differential forms of degree k,

$$
\begin{aligned}
(d\omega)(X_1,\ldots,X_{k+1}) = & \sum_i (-1)^{i-1} X_i \cdot \omega(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}) \\
& + \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{k+1}),
\end{aligned}
$$

where the hat over X_i means that this variable is deleted from the sequence.

Next we define the *interior product* of a vector field X and a k form ω.

$$(i_X\omega)(X_1,\ldots,X_{k-1}) = \omega(X,X_1,\ldots,X_{k-1}).$$

Note that $i_X\omega$ is a form of degree $k-1$. For example, when $\omega \in \Omega^1(M)$, then $i_X\omega$ is simply the function $\omega(X)$. If $\omega = \frac{1}{2}\omega^{ij} dx_i \wedge dx_j$ then

$$
\begin{aligned}
(i_X\omega)(Y) = \omega(X,Y) &= \frac{1}{2}\omega^{ij}(X_iY_j - X_jY_i) \\
&= Y_i\omega^{ji}X_j,
\end{aligned}
$$

by $\omega^{ij} = -\omega^{ji}$. Thus $i_X\omega = \theta$ with $\theta^i = X_j\omega^{ji}$. In general, for $\omega = \frac{1}{k!}\omega^{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ we have

$$(i_X\omega)^{i_1\ldots i_{k-1}} = X_j\omega^{ji_1\ldots i_{k-1}}.$$

For any smooth map $f : M \to N$ we define *the pull-back* operator $f^* : \Omega^k(N) \to \Omega^k(M)$ by

$$(f^*\omega)_x(v_1,\ldots,v_k) = \omega_{f(x)}(T_xf \cdot v_1,\ldots,T_xf \cdot v_k)$$

for $v_1,\ldots,v_k \in T_xM$. In terms of local coordinates x_i on M and y_i on N we have

$$(f^*\omega)^{i_1\ldots i_k}(x) = \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_k}}{\partial x_{i_k}} \omega^{j_1\ldots j_k}(y).$$

In the case when $M = N$ this gives us again the coordinate transformation rule of differential forms.

Exercise Show that $f^*(d\omega) = d(f^*\omega)$ and $f^*(\omega \wedge \theta) = (f^*\omega) \wedge (f^*\theta)$ for all differential forms ω, θ and a smooth map f.

The pull-back of a form $h \in C^\infty(M) = \Omega^0(M)$ is simply the composed function $f^*h = h \circ f$.

Finally we define *the Lie derivative* of a $k-$ form ω in the direction of a vector field X as the $k-$form $\mathcal{L}_X\omega$,

$$(\mathcal{L}_X\omega)(X_1,\ldots,X_k) = X \cdot \omega(X_1,\ldots,X_k) - \sum_{i=1}^{k} \omega(X_1,\ldots,[X,X_i],\ldots,X_k).$$

In terms of local coordinates,

$$(\mathcal{L}_X\omega)^{i_1\ldots i_k} = X\cdot\omega^{i_1\ldots i_k} + \sum_{\alpha=1}^{k}(\partial^{i_\alpha}X_j)\omega^{i_1\ldots i_{\alpha-1}ji_{\alpha+1}\ldots i_k}.$$

Exercise Prove the relation $\mathcal{L}_X = d\circ i_X + i_X\circ d$.

2.3 Maxwell's equations and differential forms

We arrange the Cartesian coordinates of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ as an antisymmetric 4×4 matrix,

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & +cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{pmatrix}.$$

We label the rows and columns by $\mu,\nu = 0,1,2,3$ and we set $F = \frac{1}{2}F^{\mu\nu}dx_\mu\wedge dx_\nu$.

Let ϕ be an electric scalar potential and $\mathbf{A}$ a magnetic vector potential. Then

$$\mathbf{E} = -\nabla\phi - \partial^0\mathbf{A} \text{ and } \mathbf{B} = \nabla\times\mathbf{A},$$

where $\partial^0 = \frac{1}{c}\frac{\partial}{\partial t}$ but we shall work in units with speed of light $c = 1$. Define the 1-form $A = A^\mu dx_\mu$ with $A^0 = \phi$ and $A^i = c\mathbf{A}^i$. Thus we may write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

that is, $F = dA$.

Since $d^2 = 0$ we have automatically $dF = 0$. Written in electric and magnetic field components this gives the second set of Maxwell's equations,

$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}.$$

In order to obtain a differential form expression for the first set of Maxwell's equations,

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0$$
$$\nabla\times\mathbf{B} = \mu_0\left(\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mathbf{j}\right),$$

with $\mu_0\epsilon_0 = 1/c^2$, we must first fix a metric tensor $(g_{\mu\nu})$ in space-time; this could be just the Minkowski metric diag$(1,-1,-1,-1)$ but we may take any (pseudo) Riemannian metric. Note that the second set of Maxwell's equations is intrinsic to any smooth manifold, it does not depend on the choice of metric.

We shall denote $g^{ij} = g(\partial^i, \partial^j)$ for a (pseudo) Riemannian metric $g_x : T_xM \times T_xM \to \mathbb{R}$. Recall from the relativity course that by definition the matrix (g^{ij}) is symmetric and nondegenerate. The matrix elements of the inverse matrix are denoted by (g_{ij}), so $g_{ij}g^{jk} = g^{ij}g_{jk} = \delta_{ik}$.

We define an *orientation* on a manifold M of dimension n. The manifold is oriented if we have a complete system of local coordinates such that all coordinate transformations $x'_i = x'_i(x_1, \ldots, x_n)$ satisfy the condition $\det(\frac{\partial x'_i}{\partial x_j}) > 0$.

Not every manifold can be oriented. The standard spheres S^n inherit an orientation from $\mathbb{R}^{n+1}$. The orientation on $\mathbb{R}^n$ is given by the ordered set of Cartesian coordinates $(x_1, x_2, \ldots, x_n)$. A coordinate system $(y_1, \ldots, y_n)$ on the embedded unit sphere in $\mathbb{R}^{n+1}$ is then oriented if the vectors $(\partial_y^1, \ldots, \partial_y^n, v)$ are compatible with the orientation of $\mathbb{R}^{n+1}$. Here v is the outward unit normal vector field on the sphere and compatibility means that the matrix relating the given tangent vectors to the standard basis has positive determinant. On the other hand, *the real projective plane* $P\mathbb{R}^2 = S^2/\mathbb{Z}_2 = (\mathbb{R}^3 - \{0\})/\mathbb{R}_+$, consisting of lines through the origin in $\mathbb{R}^3$, has no orientation.

A metric defines a preferred n-form on an oriented manifold, called the volume form. In terms of local oriented coordinates it is defined as

$$vol_M = |\det(g^{ij})|^{1/2} dx_1 \wedge dx_2 \wedge \ldots dx_n.$$

Let x'_i be another set of oriented coordinates. Then

$$dx'_1 \wedge dx'_2 \cdots \wedge dx'_n = \det\left(\frac{\partial x'_i}{\partial x_j}\right) dx_1 \wedge dx_2 \cdots \wedge dx_n.$$

On the other hand, $g'^{ij} = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j} g^{kl}$, and this gives

$$\det(g'^{ij}) = \det(g^{ij}) \left(\det\left(\frac{\partial x_i}{\partial x'_j}\right)\right)^2.$$

This implies that

$$|\det(g'^{ij})|^{1/2} dx'_1 \wedge dx'_2 \cdots \wedge dx'_n = |\det(g^{ij})|^{1/2} dx_1 \wedge dx_2 \cdots \wedge dx_n$$

and thus the definition of vol_M is compatible with change of oriented coordinates.

Note that the orientation is really important: If the determinant of the coordinate transformation is negative then the volume would change the sign.

A metric defines also *a duality operation* $*: \Omega^k(M) \to \Omega^{n-k}(M)$ on differential forms. In local coordinates,

$$*\omega = \theta^{i_1 i_2 \dots i_{n-k}} dx_{i_1} \wedge dx_{i_2} \dots dx_{i_{n-k}} \text{ with}$$
$$\theta^{i_1 \dots i_{n-k}} = |\det(g^{ij})|^{-1/2} \frac{1}{k!} \epsilon^{i_1 \dots i_{n-k}}{}_{j_1 \dots j_k} \omega^{j_1 \dots j_k},$$

where $\epsilon_{i_1 \dots i_n}$ is the totally antisymmetric tensor with $\epsilon_{12\dots n} = +1$ and the raising of indices is done with the help of the metric tensor as in general relativity.

Example Let $M = \mathbb{R}^4$ and g^{ij} the Minkowski metric. Then $vol_M = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$. The dual of the Maxwell 2-form $F = \frac{1}{2} F^{\mu\nu} dx_\mu \wedge dx_\nu$ is given by

$$(*F)^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\alpha\beta} F^{\alpha\beta},$$

so $(*F)^{12} = F^{03}$, and cyclic permutations of 123, and $(*F)^{01} = -F^{23}$, and cyclic permutations of 123. That is, the magnetic components of the dual are equal to $(-1)\times$ the electric components of the original and the electric components of the dual are equal to the magnetic components of the original field.

The complete set of Maxwell's equations can now be written as

$$d * F = J$$
$$dF = 0,$$

where the 3-form J is defined as $\frac{1}{3!} \epsilon^{\mu\alpha\beta\gamma} J_\mu dx_\alpha \wedge dx_\beta \wedge dx_\gamma$ with $J_0 = \rho/\epsilon_0$ and $J^k = c\mu_0 j^k$. Here ρ is the charge density and $\mathbf{j}$ is the electric current density.

2.4 de Rham cohomology

Recall that $d: \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear map with $d^2 = 0$. We set $B^k(M) = d(\Omega^{k-1}(M)) \subset \Omega^k(M)$ and $Z^k(M) = \ker d = \{\omega \in \Omega^k | d\omega = 0\} \subset \Omega^k(M)$. These are linear subspaces with the property $B^k(M) \subset Z^k(M)$, because of $d^2 = 0$. Elements of Z^k are called *closed forms* and elements of B^k are *exact forms.* We set

$$H^k(M) = Z^k(M)/B^k(M), \text{ with } k = 0, 1, 2, \dots$$

where $H^0(M) \equiv Z^0(M)$. Note that $H^k(M) = 0$ for $k > n$ since $\Omega^k(M) = 0$ for $k > n$. The vector spaces $H^k(M)$ are called the *de Rham cohomology groups* of M. In case when M is compact, one can prove that $\dim H^k(M) < \infty$ for all k.

Example $M = \mathbb{R}^3$. Since $df = 0$ for $f \in C^\infty(M) = \Omega^0(M)$ means that f is a constant function, we get $H^0(\mathbb{R}^3) = \mathbb{R}$. If $\omega = \omega^i dx_i$ satisfies $d\omega = 0$ then the vector field $(\omega^1, \omega^2, \omega^3)$ has zero curl, and we know vector analysis that there is a scalar potential f such that $\nabla f = \omega$, in other words, $df = \omega$. Thus $B^1 = Z^1$ and so $H^1(\mathbb{R}) = 0$. If $\omega = \frac{1}{2}\omega^{ij} dx_i \wedge dx_j$ is a 2-form with $d\omega = 0$ then $\mathrm{div}\,\omega = 0$ with $\omega = (\omega_{23}, \omega_{31}, \omega_{12})$. This implies that there is a vector potential $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \omega$, or in other words, $dA = \omega$, $A = A^i dx_i$. Again, $Z^2 = B^2$ and $H^2(\mathbb{R}^3) = 0$. In the same vein one can show that $H^3(\mathbb{R}^3) = 0$.

Poincare's lemma. *Let $M \subset \mathbb{R}^n$ be a star shaped open set. This means that there is a point $z \in M$ such that the line $tx + (1-t)z, 0 \le t \le 1$, belongs to M for any $x \in M$. Let ω be a closed $k-$form on M, $k > 0$. Then there exists a $(k-1)$-form θ such that $d\theta = \omega$.*

Proof. Define

$$\theta^{i_1 \dots i_{k-1}}(x) = k \int_0^1 t^{k-1}(x_j - z_j)\omega^{j i_1 i_2 \dots i_{k-1}}(tx + (1-t)z)dt.$$

We claim that $d\theta = \omega$. Now

$$\begin{aligned} d\theta &= k \int_0^1 t^{k-1}\omega^{j i_1 \dots i_{k-1}}(tx + (1-t)z)dx_j \wedge dx_{i_1} \wedge \dots dx_{i_{k-1}} dt \\ &\quad + k \int t^k (x_j - z_j)\partial^l \omega^{j i_1 \dots i_{k-1}}(tx + (1-t)z)dx_l \wedge dx_{i_1} \wedge \dots dx_{i_{k-1}} dt. \end{aligned} \tag{1}$$

The equation $d\omega = 0$ gives

$$\partial^l \omega^{j i_1 \dots i_{k-1}} \pm \text{ cyclic permutations of } l j i_1 \dots i_{k-1} \; = 0,$$

where the signs are given by the parity of the cyclic permutation. From this equation one can reduce, by setting the contraction $i_{\partial^j} d\omega$ equal to zero,

$$k\partial^l \omega^{j i_1 \dots i_{k-1}} dx_l \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} = \partial^j \omega^{l i_1 \dots i_{k-1}} dx_l \wedge \dots \wedge dx_{i_{k-1}}.$$

Note that in local coordinates

$$i_{\partial^j} d\omega^* + d i_{\partial^j}\omega^* = \mathcal{L}_{\partial^j}\omega^* = \partial^j \omega^*.$$

Inserting this to the second term I_2 on the right-hand-side of (1) we obtain

$$\begin{aligned}
I_2 &= \int_0^1 (x_j - z_j) t^k \partial^j \omega^{li_1 \dots i_{k-1}}(tx + (1-t)z) dx_l \wedge dx_{i_1} \dots \wedge dx_{i_{k-1}} dt \\
&= \int_0^1 t^k \frac{d}{dt} \omega^{li_1 \dots i_{k-1}}(tx + (1-t)z) dx_l \wedge dx_{i_1} \dots dx_{i_{k-1}} dt \\
&= -k \int_0^1 t^{k-1} \omega^{li_1 \dots i_{k-1}}(tx + (1-t)z) dx_l \wedge dx_{i_1} \wedge \dots dx_{i_{k-1}} dt \\
&+ \omega^{li_1 \dots i_{k-1}} dx_l \wedge dx_{i_1} \dots \wedge dx_{i_{k-1}}.
\end{aligned}$$

Insertion to (1) completes the proof of $d\theta = \omega$.

The above result extends (by a use of coordinates) to the case when M is a contractible subset of a smooth manifold: contractibility means that the identity map on M can be smoothly deformed to a constant map $x \mapsto X_0$ on M. Let $f_t : M \to M$ be such a contraction, $f_0(x) = x_0$ and $f_1(x) = x$, $0 \le t \le 1$. Then one can repeat the proof but with the straight lines $t \mapsto tx + (1-t)z$ replaced by $t \mapsto f_t(x), z = x_0$, see Nakahara, section 6.3, for details.

Example 1 Let $M = S^1$. The 1-form $d\phi$ is closed but $d\phi \neq df$ for any smooth function f on S^1. Note that the polar angle ϕ is not a function on S^1 since it is nonperiodic. Any 1-form on S^1 is given as $f(\phi)d\phi$ for some periodic function f of ϕ. The integral of f over the interval $[0, 2\pi]$ gives a real number λ_f. If $\lambda_f = \lambda_g$ for any two functions f, g then we can write $f - g = h'$ for a periodic function h, that is, $fd\phi - gd\phi = dh$. It follows that the cohomology classes $[f] \in H^1(S^1)$ are parametrized by the integral λ_f and so $H^1(S^1) = \mathbb{R}$.

Example 2 On the unit sphere S^2 the area form is given as $\omega = \sin\theta d\theta \wedge d\phi$ in spherical coordinates. Locally, $\omega = d(-\cos\theta d\phi) = d(-\phi \sin\theta d\theta)$. Note that the first expression becomes singular at the poles $\theta = 0, \pi$ whereas the second is nonperiodic in the coordinate ϕ. One can prove that $H^2(S^2) = \mathbb{R}$ and that the cohomology classes are parametrized by the integral of the 2-form over S^2. In general, it is known that $H^k(S^n) = 0$ for $1 \le k \le n-1$ and that $H^0(S^n) = \mathbb{R} = H^n(S^n)$.

Example 3 $H^1(S^1 \times S^1) = \mathbb{R}^2$ (basis of 1-forms $d\phi_1, d\phi_2$) and $H^2(S^1 \times S^1) = \mathbb{R}$, basis $d\phi_1 \wedge d\phi_2$.

Let M be a smooth oriented manifold of dimension n. We fix an atlas of coordinate neighborhoods compatible with the given orientation. Let $x_1, \dots, x_n$ be local coordinates on an open set $U \subset M$. Asssume that $f \in C^\infty(M)$ is such that $f(x) = 0$ when x is outside of a compact subset K of U. Then $\omega = f(x)dx_1 \wedge dx_2 \cdots \wedge dx_n$ is a $n-$ form on M. We **define** the integral

$$\int \omega = \int f(x)dx_1 dx_2 \dots dx_n,$$

as the ordinary Riemann integral in $\mathbb{R}^n$.

Let us assume that we have a *locally finite* atlas (U_α, ϕ_α). This means that for any $x \in M$ there is an open neighborhood V of x such that V intersects only a finite number of the sets U_α. A space which has a locally finite cover is said to be *paracompact.* In fact, any finite-dimensional manifold is paracompact. A locally finite atlas has a subordinate *partition of unity.* That is, there is a family of smooth nonnegative functions $\rho_\alpha : M \to \mathbb{R}$ such that

(1) $\text{supp}\rho_\alpha \subset U_\alpha$

(2) $\sum_\alpha \rho_\alpha(x) = 1$ for all $x \in M$.

The support supp$f$ of a function f is defined as a closure of the set of points x for which $f(x) \neq 0$.

Let $\omega \in \Omega^n(M)$. we define

$$\int_M \omega = \sum_\alpha \int \rho_\alpha \omega,$$

and we apply the previous definition to each term on the right-hand-side. The integral converges always when M is compact.

Exercise Show that the above definition does not depend on the choice of the partition of unity or of the locally finite atlas.

Next we want to define the integral of a form $\omega \in \Omega^k(M)$ over a parametrized $k-$ surface for arbitrary $0 \le k \le n$.

A *standard k-simplex* in $\mathbb{R}^k$ is the subset

$$\sigma_k = \{(x_1, \dots, x_k) \in \mathbb{R}^k | \sum x_i \le 1, x_j \ge 0\}.$$

So σ_0 is just a point, σ_1 is the unit interval, σ_2 is a triangle, etc.

A singular k-simplex is any smooth map $s_k : \sigma_k \to M$. A *k-chain* is a formal linear combination $\sum a_\alpha s_{k,\alpha}$, with $a_\alpha \in \mathbb{R}$ and each $s_{k,\alpha}$ is a singular $k-$ simplex.

We define an affine map $F_k^i : \sigma_{k-1} \to \sigma_k$ where $i = 0, 1, \ldots, k$. Note that the subset of points in σ_k with the coordinate $x_i = 0$ can be naturally identified as a $k-1$ simplex σ_{k-1} for $1 \leq i \leq k$. This defines the map (as an identity map) for $i = 1, 2 \ldots, k$. The remaining map F_k^0 sends the $(k-1)$-simplex σ_{k-1} to the face of the k-simplex which is not parallel to any of the coordinate axes. The map is completely fixed by requiring it to be affine and *compatible with the orientations,* and such that the origin of σ_{k-1} is mapped to the vertex of σ_k lying on the first coordinate axes, and the vertex of σ_{k-1} lying on the i :th coordinate axes is mapped to the vertex of σ_k on the $(i+1)$:th coordinate axes, for $i = 1, 2, \ldots, k-1$. See the picture below.

The *boundary* of a singular k-simplex $s_k : \sigma_k \to M$ is the singular k-chain defined as

$$\partial s_k = \sum_{i=0}^{k} (-1)^i s_k \circ F_k^i.$$

we extend the definition, by linearity, to the space C_k of singular k-chains, $\partial : C_k \to C_{k-1}$.

Theorem. $\partial^2 = 0$.

Proof. We first observe that

$$F_k^i \circ F_{k-1}^j = F_k^j F_{k-1}^{i-1}, \text{ for } j < i.$$

Let $s = \sum_\alpha a_\alpha s_{k,\alpha} \in C_k$. Then

$$\begin{aligned}
\partial^2 s &= \partial \sum_\alpha a_\alpha \sum_{i=0}^{k} (-1)^i s_{k,\alpha} \circ F_k^i \\
&= \sum_\alpha a_\alpha \sum_{i=0}^{k} (-1)^i \sum_{j=0}^{k-1} s_{k,\alpha} \circ F_k^i \circ F_{k-1}^j (-1)^j \\
&= \sum_\alpha a_\alpha \left(\sum_{0\le i\le j\le k-1} (-1)^{i+j} s_{k,\alpha} F_k^i \circ F_{k-1}^j \right. \\
&\quad \left. + \sum_{0\le j<i\le k} (-1)^{i+j} s_{k,\alpha} F_k^i \circ F_{k-1}^j \right) \\
&= \sum_\alpha a_\alpha \left(\sum_{0\le i\le j\le k-1} (-1)^{i+j} s_{k,\alpha} F_k^i \circ F_{k-1}^j \right. \\
&\quad \left. + \sum_{0\le j<i\le k} (-1)^{i+j} s_{k,\alpha} F_k^j \circ F_{k-1}^{i-1} \right).
\end{aligned}$$

Relabel $i \mapsto j, j \mapsto i-1$ in the first term of right-hand-side of the last equality; then the terms cancel.

A *cycle* is a singular chain s such that $\partial s = 0$. A *boundary* is a singular chain b such that $b = \partial s$ for some singular chain s. Denote by Z_k the space of k-cycles and by B_k the space of k-boundaries. Finally, the *singular k-homology group* is the space

$$H_k(M) = H_k(M, \mathbb{R}) = Z_k(M)/B_k(M).$$

Sometimes one considers also the homology group $H_k(M, \mathbb{Z})$ which is defined as the real homology group but one restricts to integral linear combinations of the singular k-simplexes.

Exercise Show that $H_0(M)$ is isomorphic with $\mathbb{R}^k$, where k is the number of path connected components of M.

The homology groups H_k of contractible manifolds vanish for $k > 0$, so in particular $H_k(\mathbb{R}^n) = 0$ for $k > 0$. On the other hand, $H_n(S^n) = \mathbb{R}$ but $H_k(S^n) = 0$ for $0 < k < n$.

We define the integral of a k-form over a singular k-chain $s = \sum_\alpha a_\alpha s_{k,\alpha}$,

$$\int_s \omega = \sum_\alpha a_\alpha \int_{\sigma_k} s_{k,\alpha}^* \omega.$$

Each of the integral on the right is an ordinary Riemann integral of a smooth function defined in the standard simplex $\sigma_k \subset \mathbb{R}^k$, after writing each of the pull-back forms as $f(x)dx_1 \wedge \dots dx_k$.

Theorem. *(Stokes' theorem)*

$$\int_s d\omega = \int_{\partial s} \omega$$

for any $\omega \in \Omega^{k-1}(M)$ and for any singular k-chain s.

Proof. By linearity, it is sufficient to give the proof for a single singular k-simplex s_k. But in this case a typical term in $s_k^*\omega$ can be written as

$$s_k^*\omega = \sum_{j=1}^{k} b_j(x)dx_1 \wedge \dots d\hat{x}_j \wedge \dots dx_k(-1)^{j-1}$$

for some smooth functions b_j. Then

$$d(s_k^*\omega) = s_k^*(d\omega) = \sum(\partial^j b_j)dx_1 \wedge \dots \wedge dx_k = f(x)dx_1 \wedge \dots \wedge dx_k.$$

We can now apply to familiar Gauss' theorem for vector fields in $\mathbb{R}^k$,

$$\int_{\sigma_k} \partial^j b_j dx_1 \dots dx_k = \int_{\partial\sigma_k} \mathbf{b} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the outward normal vector field on σ_k and dS is the Euclidean area measure on the surface $\partial\sigma_k$ of the k-simplex. But the right-hand-side of the equation is equal to the integral $\int_{\partial\sigma_k} s_k^*\omega$, which proves the theorem.

We have a pairing $H_k(M) \times H^k(M) \to \mathbb{R}$ which is given as

$$< [s], [\omega] >= \int_s \omega.$$

Because of Stokes' theorem the right-hand-side does not depend on particular representatives of the (co)homology classes, i.e., if $s - s'$ is a boundary and $\omega - \omega'$ is a coboundary then

$$\int_s \omega = \int_{s'} \omega'.$$

For compact oriented manifolds one can prove that the pairing is nondegenerate, i.e., if $< [s], [\omega] >= 0$ for all $[\omega]$ (resp. for all $[s]$) then $[s] = 0$ (resp. $[\omega] = 0$).

There is a more refined version of Stokes' theorem (which we are not going to prove). This uses the idea of a *closed submanifold with boundary.* A manifold M with boundary is defined using the half space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n \geq 0\}$ as a model instead of the vector space $\mathbb{R}^n$. That is, M should be equipped with a cover by open sets U and coordinate maps $\phi : U \to \mathbb{R}^n_+$ which are homeomorphism to open subsets of the half space. The coordinate transformations $\phi \circ \psi^{-1}$ are again required to be smooth in their domain of definition. Note that the derivative in the x_n direction at the boundary points $x_n = 0$ is only defined to the positive direction.

Example The closed unit ball $B^n = \{x \in \mathbb{R}^n | ||x|| \leq 1\}$ is a manifold with boundary. The set of boundary points is the manifold S^{n-1}.

Let $N \subset M$ be an oriented manifold with boundary (dimension n) embedded in M. Its boundary ∂N is a manifold of dimension $n-1$. Let $\omega \in \Omega^{n-1}(M)$. Then one can prove

$$\int_N d\omega = \int_{\partial N} \omega.$$

Note that the integral on the left is an integral of a n-form over a manifold of dimension n (and this we have already defined) and on the right we have an integral of a $(n-1)$-form over a manifold of dimension $n-1$.

Additional reading: Nakahara: 5.4, 5.5, and Chapter 6
Chern, Chen, and Lam: Chapters 2 and 3