Introduction to Partial Differential Equations

Department of mathematics and statistics, University of Helsinki

Lectures given in Spring 2011 by Petri Ola

Introduction

This course in an introduction to the theory of partial differential equations. We try to keep the necessary prerequisites to minimum, and assume just a working knowledge of differential equations and linear algebra on a first year level, as well as the basics of calculus of several variables. We will use elementary Fourier-analytic methods but intend to prove the basic facts needed. Some knowledge of Hilbert-spaces and function theory would be useful, but not absolutely necessary.

The study of partial differential equations and their solutions is an old and central field of mathematics, and it interacts with several other sciences as well as with other fields of mathematics. Let's consider some examples.

0.1. Analytic functions of one complex variable.

Consider a differentiable function $f : U \to \mathbb{C}$, where U is a domain of the complex plane $\mathbb{C}$. As those of you who have taken the course on function theory know, the existence of a *complex derivative*

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = f'(z)$$

in U is equivalent with the real part u and imaginary part v of f satisfying the Cauchy–Riemann equations

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v \quad \text{in } U.$$

This is a *system of partial differential equations*. However, it can be written in a more compact form by introducing the so-called $\overline{\partial}$–operator $\overline{\partial} = (\partial_x + i\partial_y)/2$. Then the Cauchy–Riemann equations hold if and only if $\overline{\partial} f = 0$. This is a linear first order partial differential equation (with complex coefficients). What can we learn from this? One answer is that analytic functions have several useful properties, like the Cauchy integral formula giving the value of an analytic f at a point z inside a domain, say a circle, in terms of values of f on the boundary. Also, one knows that if an analytic function vanishes in an open subset, then it vanishes in U. We'll see that similar kind of results can actually be proved for much more general equations.

0.2. Wave propagation

Consider a string connecting points 0 and l in $\mathbb{R}$. We assume that the thickness of the string is negligible, and that it has a uniform structure. Assume that the string is initially at rest, and at time $t = 0$ it is given an instantaneous pinch that causes it to vibrate. Let $u(x,t)$ be the vertical displacement of the point x at time t. If we assume that the initial perturbation is weak enough so that the all nonlinear terms can be neglected we see that the displacement satisfies the *wave equation*

$$\partial_{tt}u(x,t) = c^2\partial_{xx}^2 u(x,t), \quad 0 < x < l,\, t > 0,$$

where c is a positive constant depending on the material properties of the string. As we will see later it is natural to interpret c as the speed with which perturbations, or waves, propagate along the string. We would like to have a formula telling what the solution will be at any given time. Notice that we have to also specify something about the behavior of the end points of the string - it is easy to believe that a string whose end points are free to move behaves in different manner than a string whose end points are kept fixed.

0.3. Heat propagation

Consider a rod of some heat conducting material of length l. As above let's assume that it is placed in $\mathbb{R}$ so that the end points are 0 and l. One can again show that if the rod is homogeneous and infinitely thin the temperature of the point x, denoted by $v(x,t)$, satisfies the *heat equation*

$$\partial_t v(x,t) = \rho\partial_{xx}v(x,t), \quad 0 < x < l,\, t > 0,$$

where ρ is a positive constant depending on heat conducting properties of the rod. As we will see solutions of heat equation behave in a very different manner compared to solutions of the wave equation. For example consider the following. If u is a solution of the wave equation, then also $\widetilde{u}(x,t) = u(x,-t)$ will solve it. So, in the wave equation we can reverse time. However, if v solves heat equation, $\widetilde{v}(x,t) = v(x,-t)$ will solve the so called *backwards heat equation*

$$\partial_t\widetilde{v}(x,t) = -\rho\partial_{xx}\widetilde{v}(x,t), \quad 0 < x < l,\, t > 0,$$

which has lots of unpleasant properties, unlike the original heat equation.

0.4. Option pricing.

Consider a *call option* C on a stock S, with *expiry* $T > 0$ and *strike* K. This means that the issuing bank gives the holder of the option the possibility to purchase the stock S at time T at a fixed price K. Of course, it makes sense to call the option only if the price of the stock at time T, $S(T)$, is bigger that the strike K. Then the holder of the option can immediately sell the stocks on the market and make a profit $S(T) - K$, neglecting possible handling fees. However, the bank might not want keep the option in its own balance sheet but wants to sell it at time $0 < t < T$ to some other institution. To do that, it would be nice to have a method to price options. Note that $S(t)$ is a random variable, and it is not clear what kind of statistics it obeys. In 1973 F. Black and M. Scholes introduced a method to price options[1] They assumed that the stock price S follows a stochastic process called *Brownian motion*, and they were able to prove that the option price $C(S,t)$ satisfies a partial differential equation, called *Black–Scholes–equation*,

$$\frac{\partial C(S,t)}{\partial t} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} = rC.$$

Here r is the risk–free money–market account interest rate, and σ is a function of t and S called *volatility*. You can think of σ^2 as the variance parameter of S. Note that this looks like a backwards heat equation with a first order term added. Should we be worried? Actually no, since we do not really know a'priori the value of C at time $t = 0$. However, we know that

$$C(S(T),T) = \max\{S(T) - K, 0\}.$$

Hence we know the end value and we want to solve $C(S(t),t)$ backwards in time. This one can do, and we will return to this later in the course.

0.5. Harmonic functions

Let's go back to example 0.1 and consider the real part u of an analytic function f. We can compute using the Cauchy-Riemann equations[2]

$$(\partial_{xx} + \partial_{yy})u(x,y) = \partial_x(\partial_x u) + \partial_y(\partial_y u) = \partial_x(\partial_y v) + \partial_y(-\partial_x v) = 0$$

We call $\Delta = \partial_{xx} + \partial_{yy}$ the *Laplace operator*, and a twice differentiable function g is *harmonic* if $\Delta g = 0$. Similarly, for functions f defined in open sets of $\mathbb{R}^n$ we define

$$\Delta f = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right) f.$$

This operator is important in many other connections besides function theory. For example, the higher dimensional analogues of the wave– and heat–equations are respectively

$$\partial_{tt} u(x,t) = c^2 \Delta u(x,t), \quad x \in \mathbb{R}^n, t > 0,$$

and

$$\partial_t v(x,t) = \rho \Delta v(x,t), \quad x \in \mathbb{R}^n,\ t > 0.$$

Assume that the solutions of these equations are independent of time, i.e. they are static. For example, in the case of the heat equation it is intuitively easy to believe that if the heat sources are turned off after a finite time and that the system is well insulated, it settles into a static limit as $t \to \infty$. Then both these static solutions are actually harmonic functions of x.

[1] F.Black, M.Scholes: The Pricing of options and corporate liabilities, *Journal of Political Economy, 81 (637–654), 1973.*

[2] Note that we are using the fact that analytic functions are infinitely differentiable to change the order of differentiation

0.6. Electrostatics.

Let's consider an electric potential u in some open set Ω of $\mathbb{R}^3$. By Ohm's law the electric current is then [3]

$$I(x) = -\sigma(x)\nabla u(x).$$

where $\sigma(x)$ is the *conductivity* at the point x. Assume now that there are no current sources or sinks in Ω. Then by the definition of the divergence $\nabla \cdot I = 0$ in Ω and we obtain the *conductivity equation*

$$\nabla \cdot \sigma(x)\nabla u(x) = 0 \quad \text{in } \Omega,$$

at least if we can differentiate $\sigma(x)\nabla u(x)$. The *Dirichlet problem* associated to the conductivity equation is to determine $u(x)$ at all points in Ω given that we know values of the potential u on the boundary $\partial\Omega$. The *Neumann problem* is to determine $u(x)$ at all points in Ω given that we know normal components of the current $-\langle \nu, \sigma(x)\nabla u\rangle$ on the boundary. Both Dirichlet– and Neumann problems are traditionally called *direct problems.* In many applications[4] on is actually more interested in solving the corresponding *inverse problem*: Assume that we do **not** know values of σ in Ω, but we can place any potential distribution $u|_{\partial\Omega} = f$ one the boundary and then we can measure the normal component of the induced electric current. Note that we are measuring a large family solutions on the boundary, but we know nothing about their behavior in Ω, since we do not know the conductivity σ in Ω. Can we determine the values of σ in Ω? This problem was first formulated by Alberto Calderón (14.9.1920–16.4.1998), one of the greatest harmonic analysts of the 20th century.

0.7. Literature

There are several excellent books you can consult. These notes will not cover everything you should know - they simply contain the material that I can teach in one semester. Below is a selection of some favorites of mine[5]:

- D. Colton: *Partial Differential Equations, An Introduction*, Random House 1988. Great book, contains a nice introduction to linear equations of degree two. Has also chapters on integral equations and scattering theory.
- R. Courant and D. Hilbert: *Methods of Mathematical Physics, volume II.* A Readable and exciting classic - has a wealth of information on classical theory, both linear and non linear.
- L.C. Evans: *Partial Differential Equations*, AMS 1998. A comprehensive and modern introduction to the field. An excellent book!

[3] This is just a higher dimensional analogue of the Ohm's law you have learned in High School: I = U/R, when we define the *conductivity* σ as the inverse of the (non–zero) resistivity R

[4] This is the fundamental mathematical question behind a medical imaging modality called *Electrical Impedance Tomography.* Besides medicine, this problem is also important in many engineering applications, like detecting cracks in concrete structures, or monitoring the state of some industrial processes.

[5] As you notice, all books seem to have nearly identical titles.

- L. Hörmander: *The Analysis of Linear Partial Differential Operators, vol. I - IV*, Springer 1983 - 1985. Not really an introduction, but an excellent source for more advanced material on linear equations. The first volume contains the best introduction to distributions (i.e. generalized functions) I know. The second volume has a very elegant chapter on mathematical foundations of scattering theory. The third is an introduction to micro–local analysis and pseudodifferential operators, and finally the fourth volume concentrates on Fourier integral operators.

- F. John: *Partial Differential Equations*, Springer. A classic, but not easy, introduction to the field.

- Y. Pinchover – J. Rubinstein: *An Introduction to Partial Differential Equations*, Cambridge University Press 2005. A more recent and a very readable book.

- S.L. Sobolev: *Partial Differential Equations of Mathematical Physics*, Pergamon Press (there is also a Dover edition). An excellent introduction by one of the founders of the modern theory of PDE's.

1. Equations of first order in two independent variables

Summary.

It is natural to start with first order equations. First of all, their theory has a nice geometric flavour. Also, they admit quite general solvability results which are lacking in many higher order problems. In these lectures we will only consider equations in two independent variables in order to keep the central ideas as clearly in view as possible. Naturally, both the methods and results carry over to higher dimensions. Also, we restrict ourselves to equations with *real* coefficients. Results will be dramatically different when complex coefficients are allowed[6]. In what follows, partial derivatives of a function f in x and y variables are denoted by f_x and f_y respectively.

1.1. Linear equations.

Assume $\Omega \subset \mathbb{R}^2$ is an open set. *A linear first order equation* is of the form

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c_0(x,y) + c_1(x,y)u(x,y), \quad (x,y) \in \Omega, \tag{1.1.1}$$

where a, b, c_0, c_1: $\Omega \to \mathbb{R}$ are given *coefficient functions*, and we want to find $u \in C^1(\Omega)$ solving (1.1.1). This is called 'first order', since it contains only first order derivatives of

[6] See for example F. John: *Partial Differential Equations*, Ch. 8

the unknown u, and it is linear, since its dependence on the unknown u is linear. More precisely, if we introduce a *partial differential operator*

$$L : C^1(\Omega) \to C(\Omega), \quad Lf = af_x + bf_y - c_1 f,$$

then the map L is linear, and (1.1.1) can be formulated as

$$Lu = c_0.$$

We want to find conditions for coefficients that guarantee the existence of a solution, at least a local existence near some point $(x_0, y_0) \in \Omega$, and if possible to find conditions that guarantee the uniqueness of the solution.

1.1.1. A simple example. Consider the equation

$$u_x(x,y) = c_1 u(x,y) + c_0(x,y), \quad (x,y) \in \mathbb{R}^2, \tag{1.1.2}$$

i.e. in (1.1.1) we take $a(x,y) = 1$, $b(x,y) = 0$ and $c_1(x,y) = c_1$, a constant. This is just an ordinary differential equation in x with y as a parameter. Assuming that $t \mapsto c_0(t,y)$ is integrable over all bounded intervals for all y we can solve this to get

$$u(x,y) = e^{c_0 x}\left(\int_0^x e^{-c_0 t} c_0(t,y)\, dt + \phi(y)\right),$$

where ϕ is an arbitrary function of y. This formula gives all solutions which are differentiable in x. Furthermore, we see that the only cause of non–uniqueness is ϕ, and to fix the solution uniquely it is enough to fix ϕ uniquely. This can be achieved by giving the value of u on the y–axis,

$$u(0,y) = f(y), \quad y \in \mathbb{R}, \tag{1.1.3}$$

Then $f(y) = u(0,y) = \phi(y)$, and the solution to the *initial value problem* (1.1.2), (1.1.3) is given by

$$u(x,y) = e^{c_1 x}\left(\int_0^x e^{-c_1 t} c_0(t,y)\, dt + f(y)\right),$$

Examples of other kinds of initial conditions are in exercises.

1.1.2. A constant coefficients equation. Motivated by the previous example, let's consider a *linear constant coefficient equation*

$$au_x(x,y) + bu_y(x,y) = c_0(x,y) + c_1 u(x,y), \quad (x,y) \in \mathbb{R}^2. \tag{1.1.4}$$

Here a, b and c_1 are constants. Let $v = (a,b)$. Using the directional derivative we can then write (1.1.4) as

$$\partial_v u = c_0 + c_1 u. \tag{1.1.5}$$

This of the form studied in 1.1.1 once we perform a linear change of variables. Let

$$x = a\xi + b\eta,\ y = b\xi - a\eta.$$

Then, if $a^2 + b^2 \neq 0$, which we assume since otherwise there are no derivatives in the equation, we can solve to get

$$\xi = (a^2 + b^2)^{-1}(ax + by), \quad \eta = (a^2 + b^2)^{-1}(bx - ay),$$

and by the chain rule

$$\partial_\xi = \frac{\partial x}{\partial \xi}\partial_x + \frac{\partial y}{\partial \xi}\partial_y = a\partial_x + b\partial_y = \partial_v.$$

Hence, if by capital letters we denote functions in (ξ, η)–variables, i.e. $F(\xi, \eta) = f(a\xi + b\eta, b\xi - a\eta) = f(x, y)$, we have that (1.1.5) is equivalent with

$$\partial_\xi U(\xi, \eta) = C_0(\xi, \eta) + c_1 U(\xi, \eta).$$

This has a general solution, again assuming the integrability of $t \mapsto C_0(t, \eta)$ for all η,

$$U(\xi, \eta) = e^{-c_1\xi}\left(\int_0^\xi e^{-c_1 t} C_0(t, \eta)\, dt + \Phi(\eta)\right).$$

To get an expression for u in original variables we make an observation. By dividing the equation (1.1.4) by $a^2 + b^2$ we do not change the solutions, but get an equation of the same form where v is a unit vector. So, assume that $a^2 + b^2 = 1$. Then substituting back we get

$$u(x, y) = e^{-c_1(ax+by)}\left(\int_0^{ax+by} e^{-c_1 t} c_0(x(t), y(t))\, dt + \Phi(bx - ay)\right).$$

where $(x(t), y(t))$ is determined by the conditions

$$ax(t) + by(t) = t, b(x(t) - x) - a(y(t) - y) = 0,$$

A natural place to pose an initial condition is to do it on the line $ax + by = 0$, i.e. on the set $\{\xi = 0\}$. Assume that $U(0, \eta) = F(\eta)$. Then $F(\eta) = \Phi(\eta)$ and thus the solution of the initial value problem is

$$u(x, y) = e^{-c_1(ax+by)}\left(\int_0^{ax+by} e^{-c_1 t} c_0(x(t), y(t))\, dt + F(bx - ay)\right).$$

1.1.3. A geometric formulation of equation (1.1.1). Let's now return to the general linear equation (1.1.1), and assume that $u \in C^1(\Omega)$ is a solution. *For the rest of this section we make the simplifying assumption* $c_1 = 0$. We will remove this restriction later. Geometrically one can consider the map $(x, y) \mapsto u(x, y)$ as a local representation of a surface in $\mathbb{R}^3$. A normal direction to the surface at point $(x, y, u(x, y))$ is

$$n(x, y) = (\partial u_x(x, y), \partial_y u(x, y), -1),$$

Assume now, as already mentioned, that $c_1 = 0$ i.e u is a solution of the homogenous problem

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) - c_0(x, y) = 0, \quad (x, y) \in \Omega, \tag{1.1.6}$$

Consider the vector field determined by the coefficients,

$$t(x, y) = (a(x, y), b(x, y), c_0(x, y)).$$

Then (1.1.6) can be written as

$$\langle t(x, y), n(x, y)\rangle = 0. \tag{1.1.7}$$

So, a geometric way to interpret the homogeneous problem is as follows: *Find a surface whose normal is always orthogonal to the coefficient vector field* $t(x, y)$. This will be the guiding idea behind the construction of the solution.

1.1.4. Characteristic curves. We can formulate the above geometric condition by saying that the coefficient field $t(x, y)$ should always be tangent to the solution surface. Since finding curves is easier than finding surfaces, we start by looking for differentiable curves in $\mathbb{R}^3$ which are always tangent to $t(x, y)$. Consider a differentiable curve

$$I \ni t \to \gamma(t) = (x(t), y(t), z(t))$$

in $\mathbb{R}^3$. Here $I \subset \mathbb{R}$ is an interval. This curve is tangent to t if[7]

$$\gamma'(t) = t(\gamma(t)). \tag{1.1.8}$$

This equation is called a *characteristic equation* of the linear system (1.1.6). In components it takes the form

$$\begin{aligned} x'(t) &= a(x(t), y(t)) \\ y'(t) &= b(x(t), y(t)) \\ z'(t) &= c_0(x(t), y(t)) \end{aligned} \tag{1.1.9}$$

As you know from the elementary course on ordinary differential equations, to fix the solution uniquely it suffices to fix the value at some point t_0. So, given $(x_0, y_0) \in \Omega$ and $z_0 \in \mathbb{R}$ we demand at $t_0 \in I$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0. \tag{1.1.10}$$

i.e. the solution curve passes through the point (x_0, y_0, z_0). An application of the local version of the existence and uniqueness theorem of ordinary differential equations immediately gives the following:

[7]This may look not general enough, since one in fact requires only $\gamma'(t) = \lambda(t)t(\gamma(t))$ for some non-vanishing continuous scalar valued function $\lambda(t)$ However, if we make the bijective change of varibles $s = \Lambda(t)$, where $\Lambda'(t) = \lambda(t)$, we get that

$$\frac{d(\gamma \circ \Lambda^{-1})}{ds} = \gamma'(\Lambda^{-1}(s))(\Lambda^{-1})'(s) = t(\gamma \circ \Lambda^{-1}(s))$$

since $(\Lambda^{-1})'(s) = 1/\Lambda'(t)$ when $s = \Lambda(t)$. Hence we can always replace γ by its reparametrization $\gamma \circ \Lambda^{-1}$, and we are back at equation (1.1.7)

1.1.5 LEMMA. *Assume that a and b are Lipshcitz–continuous in both variables, and that* c_0 *is a locally integrable function. Then for any* $(x_0, y_0) \in \Omega$ *and* $z_0 \in \mathbb{R}$ *there is a unique solution to* (1.1.9, 1.1.10) *defined in a neighborhood of* t_0.

Proof. Note that it is enough to solve the autonomous system

$$x'(t) = a(x(t), y(t)), \quad y'(t) = b(x(t), y(t)) \tag{1.1.11}$$

with initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{1.1.12}$$

for then $z(t)$ is just

$$z(t) = z_0 + \int_{t_0}^{t} c_0(x(t'), y(t'))\, dt'.$$

The unique local solvability of (1.1.11, 1.1.12) follows from the local existence and uniqueness theorem for ordinary differential equations. □

1.1.6. Example. Consider the equation

$$-yu_x + xu_y = 1.$$

Now $a(x, y) = -y$, $b(x, y) = x$ and $c_0(x, y) = 1$. Let's find the characteristic curve with initial conditions $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$. For x and y we have to solve

$$x'(t) = -y(t), \quad y'(t) = x(t).$$

Hence for x we get the second order equation

$$x''(t) = -y'(t) = -x(t),$$

which has the general solution

$$x(t) = C_1 \sin t + C_2 \cos t,$$

and then

$$y(t) = -x'(t) = -C_1 \cos t + C_2 \sin t.$$

Using the initial conditions we get

$$x_0 = x(0) = C_2, \quad y_0 = y(0) = -C_1.$$

The projection to the xy–plane of the characteristic curve is

$$t \mapsto (-y_0 \sin t + x_0 \cos t, x_0 \sin t + y_0 \cos t).$$

Finally we can integrate to get $z(t)$,

$$z(t) = z_0 + \int_0^t ds = z_0 + t.$$

Hence the characteristic curve is defined for all t, and is given by

$$t \mapsto (-y_0 \sin t + x_0 \cos t, x_0 \sin t + y_0 \cos t, z_0 + t). \tag{1.1.13}$$

1.1.7. Integral surfaces. A surface of the form $z = u(x,y)$ is an *integral surface* of (1.1.1) when u is a solution to (1.1.6). We now aim to construct the solution to (1.1.6 - or the integral surface determined by it - as a union of a family of characteristic curves. To do this we consider a curve in $\Omega \times \mathbb{R}$ given by

$$J \ni s \mapsto \Gamma(s),$$

where J is an interval of $\mathbb{R}$. Consider the characteristic system

$$\begin{aligned} x'(t) &= a(x(t), y(t)) \\ y'(t) &= b(x(t), y(t)) \\ z'(t) &= c_0(x(t), y(t)) \end{aligned} \tag{1.1.14}$$

We pose the initial condition

$$(x(0), y(0), z(0)) = \Gamma(s). \tag{1.1.15}$$

Hence, we look for a characteristic curve that at $t = 0$ hits $\Gamma(s)$. By Lemma 1.1.5 we know that a unique solution exsists, at least for t in some possibly s–dependent neighborhood $I(s)$ of 0. Denote this solution by $\gamma(\cdot, s)$, i.e.

$$\gamma(t,s) = (x(t,s), y(t,s), z(t,s)), \quad (t,s) \in I(s) \times J.$$

It is natural to think that this map defines a differentiable surface of $\mathbb{R}^3$. However, one has to be bit more careful, for this might not always be true. Also, we want to have a surface that can be written in the form $z = u(x,y)$ for some differentiable function u.

1.1.8. Example. Consider the following trivial example:

$$u_x = 1, \quad u(x,0) = h(x),$$

for some differentiable function h. The characteristic system is

$$x'(t) = 1, \quad y'(t) = 0, \quad z'(t) = 1,$$

and this has the solution

$$x(t) = t + C_1, \quad y(t) = C_2, \quad z(t) = t + C_3.$$

So, $y =$ constant. By the initial condition we want the solution to pass through points $(x, 0, h(x))$, hence $y = 0$, and thus all characteristic curves lie in the xz–plane, and they do not define a surface of the form $z = u(x,y)$. This example shows that, even for very simple equations, one has to pose some restrictions on the initial conditions.

1.1.9. Transversality condition. As a preliminary step, consider the map

$$\varphi : I(s) \times J \ni (t,s) \mapsto (x(t,s), y(t,s)), \tag{1.1.16}$$

where $(x(t,s), y(t,s), z(t,s))$ is the unique solution of the initial value problem (1.1.14 , 1.1.15). Let's compute the Jacobian of φ on the initial curve Γ:

$$\varphi'(0,s) = \begin{vmatrix} \partial x(0,s)/\partial t & \partial x(0,s)/\partial s \\ \partial y(0,s)/\partial t & \partial y(0,s)/\partial s \end{vmatrix}.$$

We can compute this using the characteristic equation and the initial conditions: From (1.1.14) we get, if we denote $\Gamma(s) = (x_\Gamma(s), y_\Gamma(s), z_\Gamma(s))$,

$$\partial x(0,s)/\partial t = a(x(0,s), y(0,s)) = a((x_\Gamma(s), y_\Gamma(s)),$$

and

$$\partial y(0,s)/\partial t = b(x(0,s), y(0,s)) = b((x_\Gamma(s), y_\Gamma(s)),$$

For the s–derivatives we use the initial conditions:

$$\partial x(0,s)/\partial s = x'_\Gamma(s), \quad \partial y(0,s)/\partial s = y'_\Gamma(s).$$

The map φ is a local diffeomorphism at $(0,s)$ if and only if $\det \varphi'(0,s) \neq 0$, i.e.

$$x'_\Gamma(s)\, b((x_\Gamma(s), y_\Gamma(s)) - y'_\Gamma(s)\, a((x_\Gamma(s), y_\Gamma(s)) \neq 0. \tag{1.1.17}$$

Vector $(-y'_\Gamma, x'_\Gamma)$ is a normal vector to the projection of the initial data curve to xy–plane, and the condition (1.1.17) means that the this normal should not be perpendicular to the xy–projection of the coefficient field, i.e. the projcetion of the coefficient field should not be parallel to the projection of the initial data curve, that is that they are *transversal.* The condition (1.1.17) is called a *transversality condition.*

1.1.10 PROPOSITION. *Consider the initial value problem* (1.1.14 , 1.1.15), *with the assumptions of Lemma* 1.1.5 *and assume that the transversality condition holds at point* $(0, s_0)$. *Then the map* $(t,s) \mapsto (x(t,s), y(t,s), z(t,s))$ *defines in some neighborhood of* $(0, s_0)$ *a parametric representation of a solution* u *of* (1.1.1) *such that* $z_\Gamma(s) = u(x_\Gamma(s), y_\Gamma(s))$.

Proof. As we saw above, the transversality condition guarantees that there are neighborhoods U of $(0, s_0)$ and V of $(x_0, y_0) = (x_\Gamma(s_0), y_\Gamma(s_0))$ such that $\varphi : U \to V, (t,s) \mapsto (x(t,s), y(t,s))$ is a diffeomorhpism. Let

$$z(t,s) = z_\Gamma(s) + \int_0^t c_0(x((t', s), y(t', s))\, dt'.$$

and define

$$u(x,y) = z(\varphi^{-1}(x,y)), \quad (x,y) \in V.$$

The surface $\{z = u(x,y)\}$ is then a union of characteristic curves, i.e. an integral surface. □

1.1.11. Continuation of the example 1.1.6. Consider the initial value poblem

$$-yu_x + xu_y = 1,\ u(x,0) = f_+(x),\ x > 0.$$

A convenient paramterization for the initial condition is

$$s \mapsto \Gamma(s) = (s, 0, f_+(s)),\ s > 0.$$

The corresponding initial value problem for the characteristic equation is

$$x'(t,s) = -y(t,s), \quad y'(t,s) = x(t,s),\ x(0,s) = s,\ y(0,s) = 0,$$

and then

$$z(t,s) = f(s) + \int_0^t dt = f(s) + t.$$

From (1.1.13) we get

$$x(t,s) = s\cos t,\ y(t,s) = s\sin t,\ z(s,t) = f_+(s) + t.$$

Notice that the parametrization is 2π–periodic in t, and since we must have $x = s\cos s > 0$ let's restrict $t \in (-\pi/2, \pi/2)$. Then

$$x(t,s)^2 + y(t,s)^2 = s^2,\ \frac{y(t,s)}{x(t,s)} = \tan t,$$

and hence

$$s = \sqrt{x^2 + y^2},\ t = \arctan y/x.$$

and the solution is

$$z = u(x,y) = f_+(\sqrt{x^2 + y^2}) + \arctan y/x,\ x > 0.$$

Similarly, let's consider the analogous problem on the left half plane,

$$-yu_x + xu_y = 1,\ u(x,0) = f_-(x),\ x < 0.$$

Now a natural choice of parameter intervals are $s < 0$ and $t \in (-\pi/2, \pi/2)$ and thus the solution is

$$z = u(x,y) = f_-(-\sqrt{x^2 + y^2}) + \arctan y/x,\ x < 0.$$

Note that the transversality condition fails across $\{x = 0\}$, so the restriction to either $x > 0$ or $x < 0$ is natural. Note also that these solutions will be continuous across $\{x = 0, y > 0\}$ if $f_+(0) + \pi/2 = f_-(0) - \pi/2$, i.e. $f_+(0) - f_(0) = -\pi$, and continuous across $\{x = 0, y < 0\}$ if $f_+(0) - \pi/2 = f_-(0) + \pi/2$, i.e. $f_+(0) - f_-(0) = \pi$.

1.2. Quasilinear equations.

Assume $\Omega \subset \mathbb{R}^2$ is an open set. *A quasilinear first order equation* is of the form

$$a(x, y, u(x, y))u_x(x, y) + b(x, y, u(x, y))u_y(x, y) = c(x, y, u(x, y)), \quad (x, y) \in \Omega, \tag{1.2.1}$$

where a, b and c: $\Omega \times \mathbb{R} \to \mathbb{R}$ are given *coefficient functions*, and we want to find $u \in C^1(\Omega)$ solving (1.1.1). This is called quasilinear, since its dependence on the derivatives of the unknown u is linear. More precisely, if we introduce a map

$$L : C^1(\Omega) \times C(\Omega) \to C(\Omega), \quad L(f, g)(x, y) = a(x, y, f)g_x + b(x, y, f)g_y - c(x, y, f),$$

then the map L is linear in the g–variable, and (1.2.1) can be formulated as

$$L(u, u) = 0.$$

Again, we want to find conditions for coefficients that guarantee the existence of a solution, at least a local existence near some point $(x_0, y_0) \in \Omega$, and if possible to find conditions that guarantee the uniqueness of the solution. The preamble 'quasi', i.e. almost, is justified since the construction of the solutions proceeds very similarly to the linear case.

1.2.1. A geometric formulation of (1.2.1).

Let's now give a geometric intepretation to equation (1.2.1), and assume that $u \in C^1(\Omega)$ is a solution. A normal direction to the surface $z = u(x, y)$ at point $(x, y, u(x, y))$ is

$$n(x, y) = (\partial u_x(x, y), \partial_y u(x, y), -1),$$

Consider the vector field in $\mathbb{R}^3$ determined by the coefficients,

$$t(x, y, z) = (a(x, y, z), b(x, y, z), c(x, y, z)).$$

Then (1.1.6) can be written as

$$\langle t(x, y, u(x, y)), n(x, y)\rangle = 0 \tag{1.2.2}$$

Hence, a geometric way to interpret the homogeneous problem is: *Find a surface whose normal is always orthogonal to the coefficient vector field* $t(x, y, u(x, y))$. The only difference compared to the linear case is that now also the vector field t depends on values of the solution u.

1.2.2. Return to the characteristic equation.

As before, we start by looking for differentiable curves in $\mathbb{R}^3$ which are always tangent to $t(x, y, z)$. So, let γ be a differentiable curve

$$I \ni t \mapsto \gamma(t) = (x(t), y(t), z(t))$$

in $\mathbb{R}^3$. Here $I \subset \mathbb{R}$ is an interval. This curve is tangent to t if

$$\gamma'(t) = t(\gamma(t)), \; t \in I. \tag{1.2.3}$$

This equation is called a *characteristic equation* of the quasilinear system (1.2.1). In components it takes the form

$$
\begin{aligned}
x'(t) &= a(x(t), y(t), z(t)) \\
y'(t) &= b(x(t), y(t), z(t)) \\
z'(t) &= c(x(t), y(t), z(t))
\end{aligned}
\tag{1.2.4}
$$

An application of the existence and uniqueness theorem of ODE's immediately gives the following:

1.2.3 Lemma. *Assume that a, b and c are Lipshcitz–continuous in all variables. Then for any $(x_0, y_0) \in \Omega$ and $z_0 \in \mathbb{R}$ there is a unique solution to* (1.2.2) *with initial values $x(t_0) = x_0$, $y(t_0) = y_0$ and $z(t_0) = z_0$ defined in a neighborhood of t_0.* □

1.2.4. Transversality condition revisited. We fix an initial data curve in $\Omega \times \mathbb{R}$ given by

$$J \ni s \mapsto \Gamma(s),$$

where J is an interval of $\mathbb{R}$. Consider the initial value problem

$$
\begin{aligned}
x'(t) &= a(x(t), y(t), z(t) \\
y'(t) &= b(x(t), y(t), z(t)) \\
z'(t) &= c_0(x(t), y(t), z(t))
\end{aligned}
\tag{1.2.5}
$$

with initial condition

$$(x(0), y(0), z(0)) = \Gamma(s). \tag{1.2.6}$$

By Lemma 1.2.3 we know that a unique solution exsists, at least for t in some possibly s–dependent neighborhood $I(s)$ of 0. Denote this solution by $\gamma(\cdot, s)$, i.e.

$$\gamma(t, s) = (x(t, s), y(t, s), z(t, s)), \quad (t, s) \in I(s) \times J.$$

As before we consider the map

$$\varphi : I(s) \times J \ni (t, s) \mapsto (x(t, s), y(t, s)),$$

and we want to find a condition implying that φ is a local diffeomorphism at $(0, s)$. From (1.2.5) we get, if we denote $\Gamma(s) = (x_\Gamma(s), y_\Gamma(s), z_\Gamma(s))$,

$$\partial x(0, s)/\partial t = a(x(0, s), y(0, s), z(0, s) = a((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)),$$

and

$$\partial y(0, s)/\partial t = b(x(0, s), y(0, s), z(0, s) = b((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)),$$

For the s–derivatives we use the initial conditions:

$$\partial x(0, s)/\partial s = x'_\Gamma(s), \quad \partial y(0, s)/\partial s = y'_\Gamma(s).$$

Hence the Jacobian of φ on the initial curve Γ is

$$\varphi'(0,s) = \begin{vmatrix} \partial x(0,s)/\partial t & \partial x(0,s)/\partial s \\ \partial y(0,s)/\partial t & \partial y(0,s)/\partial s \end{vmatrix}.$$

$$= x_\Gamma'(s)\, b((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)) - y_\Gamma'(s)\, a((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)).$$

We see that this now depends also on the initial values of the solution, and not just the set in xy–plane where the initial conditions are posed.

1.2.5 PROPOSITION. *Consider the initial value problem (1.2.5 , 1.2.6), with the assumptions of Lemma* 1.2.3*and assume that the transversality condition*

$$x_\Gamma'(s)\, b((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)) - y_\Gamma'(s)\, a((x_\Gamma(s), y_\Gamma(s), z_\Gamma(s)) \neq 0 \tag{1.2.7}$$

holds at point $(0, s_0)$*. Then the map* $(t,s) \mapsto (x(t,s), y(t,s), z(t,s))$ *defines in some neighborhood of* $(0, s_0)$ *a parametric representation of a solution* u *of (1.2.1) such that* $z_\Gamma(s) = u(x_\Gamma(s), y_\Gamma(s))$*.*

Proof. The proof is like the that of Proposition 1.1.10. The only difference is that $z(t,s)$ cannot be explicitly integrated, but it is defined as the unique solution of

$$z'(t,s) = c(x(t,s), y(t,s), z(t,s)).$$

Then $u(x,y) = z \circ \varphi^{-1}(x,y)$. □

1.2.6. Example Let's consider the initial value problem

$$-yu_x + xu_y = u^2,\ u(x,0) = f(x),\ x > 0.$$

We parametrize the initial condition as $\Gamma(s) = (s, 0, f(s))$. It is easy to see that the transversality condition holds, and the characteristic curves can be solved from

$$x'(t,s) = -y(t,s),\ x(0,s) = s,$$
$$y'(t,s) = x(t,s),\ y(0,s) = 0,$$
$$z'(t,s) = z(t,s)^2,\ z(0,s) = f(s).$$

This gives $x(t,s) = s\cos t$, $y(t,s) = s\sin t$ as before. For z we get

$$z(\tau, s) = -\frac{1}{\tau + C(s)},$$

and from the initial condition we get $C(s) = -1/f(s)$. Since $s > 0$ we have $s = \sqrt{x^2+y^2}$ and $t = \arctan(y/x)$, and hence the solution is

$$z = u(x,y) = \frac{f(\sqrt{x^2+y^2})}{1 - f(\sqrt{x^2+y^2})\arctan(y/x)}.$$

Notice that this is defined only when

$$1 - f(\sqrt{x^2+y^2})\arctan(y/x) \neq 0,$$

which always holds in a neighborhood of the x–axis if f is continuous.

1.2.7 Proposition. *In the notation of Proposition* 1.2.5*, the integral surface determined by the solutions of* (1.2.5 , 1.2.6) *gives the unique integral surface containing the initial curve* Γ *if the transversality condition* (1.2.7) *holds.*

Proof. We use the notation of proposition 1.2.5. So, let $z = U(x, y)$ be an integral surface containing the point $\Gamma(s)$ of the initial curve. Consider the map

$$v(t) = U(x(t, s), y(t, s)) - z(t, s).$$

We show that $v = 0$ in a neighborhood of $t = 0$. First, $v(0) = 0$ since $\Gamma(s)$ belongs to the integral surface U. By differentiating with respect to t we get

$$v'(t) = U_x(x(t, s), y(t, s))x'(t, s) + U_y(x(t, s), y(t, s))y'(t, s) - z'(t, s).$$

Since $(x(t, s), y(t, s), z(t, s))$ solves the characteristic system, we get

$$\begin{aligned} v'(t) = a(x(t, s), y(t, s), z(t, s))U_x(x(t, s), y(t, s)) + \\ + b(x(t, s), y(t, s), z(t, s))U_y(x(t, s), y(t, s)) - c(x(t, s), y(t, s), z(t, s)). \end{aligned}$$

Let's substitute $z(t, s) = v(t) + U(x(t, s), y(t, s))$ in the above. This gives a differential equation for v:

$$v' = a(x, y, v + U)U_x + b(x, y, v + U) - c(x, y, v + U).$$

Since U is an integral surface, $v = 0$ is a trivial solution with initial condition $v(0) = 0$. Since a, b and c satisfy a local Lipschitz condition in the third variable, this is also the only solution. Hence $v = 0$. Since this holds for all points of Γ, the two integral surfaces coincide in a neighborhood. □

1.2.8. Remark. The proof of the above proposition actually implies that the intersection of two integral surfaces will contain completely all characteristic curves having points in the intersection.

1.2.9. Shock waves via an example. Let's study the following quasilinear initial value problem,

$$u_t + u\, u_x = 0,\ u(x, 0) = f(x).$$

We will see that the form of the initial value value function f will dramatically affect the global[8] behavior of the solution u. If we parametrize the initial conditions by $\Gamma(s) = (s, 0, f(s)$, then since $a = u$, $b = 1$ and $c = 0$, the transversality condition is satisfied:

$$x'_\Gamma b - t'_\Gamma a = 1 \neq 0,$$

[8] *Global* here refers to behavior for large t, in contrast to mostly *local* claims so far proved.

hence there will always be local solution near the initial surface. The characteristic initial value problem is[9]

$$\begin{aligned} x'(\tau,s) &= z(\tau,s),\ x(0,s)=s \\ t'(\tau,s) &= 1,\ y(0,s)=0 \\ z'(\tau,s) &= 0,\ z(0,s)=f(s). \end{aligned}$$

The last equation implies that z is constant along a characteristic curve, in fact the initial condition gives

$$z(\tau,s)=f(s),\ \tau\in\mathbb{R}. \tag{1.2.8}$$

From the second condition we get

$$t(\tau,s)=\tau,\ \tau\in\mathbb{R}. \tag{1.2.9}$$

Hence the first equation gives

$$x'(\tau,s)=f(s),\ x(0,s)=s,$$

i.e. we have

$$x(\tau,s)=f(s)\tau+s,\tau\in\mathbb{R}. \tag{1.2.10}$$

Note that the characteristic curves exist for all $\tau\in\mathbb{R}$. Since the transversality condition is valid, we know that in a neighborhood of $\{t=0\}$ this defines an integral surface $z=u(x,y)$. In fact, from the last equation we get $s=x-f(s)\tau=x-ut$, and plugging this into (1.2.8) we get an implicit equation for u:

$$u=f(x-ut). \tag{1.2.11}$$

Consider a characteristic curve $\gamma(\tau,s)=(x(\tau,s),y(\tau,s),z(\tau,s))$. The xt–projection has the parametric representation

$$x(\tau,s)=f(s)\tau+s,\quad t(\tau,s)=\tau,$$

i.e. it is a line with slope $f(s)$ and intersecting the t–axis at $(s,0)$. Consider now two characteristic curves $\gamma(\cdot,s_1)$ and $\gamma(\cdot,s_2)$ with $s_1<s_2$. If $f(s_1)=f(s_2)$ their projections to xt–plane are parallel, and hence do not intersect. If $f(s_1)\neq f(s_2)$ the projections intersect at

$$p(s_1,s_2):=\left(\frac{f(s_1)s_2-f(s_2)s_1}{f(s_1)-f(s_2)},\frac{s_2-s_1}{f(s_1)-f(s_2)}\right).$$

If $f(s_1)>f(s_2)$ the intersection lies in the left half–plane $\{t<0\}$ and if $f(s_1)<f(s_2)$ the intersection lies in the right half–plane $\{t>0\}$ What is crucial is that these projections intersect! Namely, as we noticed above, the the solution is constant on a characteristic curve. On the set $x=f(s_1)t+s_1$ it takes value $f(s_1)$, and on $x=f(s_1)t+s_1$ it has

[9] Note that we have now denoted the variable along the characteristic curve by τ since t is a variable already in the equation itself.

value $f(s_2)$. Hence **it is not defined** at the intersection point, and generally we do not have global existence for solutions. One can also see this from (1.2.11) more directly: differentiating both sides with respect to x we get

$$u_x = f'(x - ut)(1 - tu_x),$$

i.e

$$u_x = \frac{f'(x - ut)}{1 + th'(x - ut)},$$

and this blows up at $t = -(h'(x-ut))^{-1}$. Note that after this intersection the characteristic curves again define an integral surface.

1.3. Fully nonlinear equations.*

1.3.1. Eikonal equation.

1.3.2. Monge cone.

1.3.3. Solutions generated as envelopes.

1.3.4. Characteristic curves.

1.3.5. Characteristic equations.

1.3.6. Plane elements and strip conditions.