Malliavin calculus, exercises 3, 23.02.2011

1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(G) \in L^2(P)$, when $G(\omega)$ is gaussian with $E_P(G) = 0$ and $E_P(G^2) = 1$.

   Here f does not need to be differentiable in any sense.

   - Show that the function
     $$Tf(t) := E_P(f(G+t)) = \int_{\mathbb{R}} f(x+t)\gamma(x)dx$$
     is classically differentiable infinitely many times with respect to x.
     Hint:
     $$\frac{1}{\varepsilon}\int_{\mathbb{R}} \big(f(x+t+\varepsilon) - f(x)\big)\gamma(x)dx = \frac{1}{\varepsilon}\int_{\mathbb{R}} f(x+t)\big(\gamma(x-\varepsilon) - \gamma(x)\big)dx$$
     A sufficient condition for
     $$\frac{d}{dt}\int \psi(x,t)\mu(dx) = \int \frac{d}{dt}\psi(x,t)\mu(dx)$$
     at $t = t_0$ is that for all t in a neighbourhood of t_0
     $$\left|\frac{d}{dt}\psi(x,t)\right| \le \rho(x)$$
     for some dominating function $\rho \in L^1(\mu)$ (this follows by Lebesgue dominating convergence theorem, see Williams probability and martingales, A16).
     Find a dominating function.
   - Compute the derivatives. Hint: remember the Hermite polymomials !
   - Compute the Taylor expansion of $Tf(t)$ around $t_0 = 0$.
   - Compute the likelihood ratio $Z(G,t) = \frac{dQ}{dP}$ of a change of measure such that under Q $G(\omega)$ is a gaussian variable with $E_Q(G) = t$ ja $E_Q(G^2) = 1$, and show that
     $$E_P(f(G+t)) = E_Q(f(G)) = E_P(f(G)Z(G,t))$$
   - Deduce that in the Taylor expansion of $Tf(t)$ around $t_0 = 0$ we can interchange the order expectation and summation.
     Hint: remember the expansion for the generating function
     $$Z(x,t) = \exp\big(tx - \frac{1}{2}t^2\big)$$

2. Let $(G_t : t \in \mathbb{N})$ a sequence of independent gaussian martingales. $H_n(x)$ denotes the n-th (un-normalized) Hermite polynomial.

- Show that

$$M_t = \sum_{s=1}^{t} H_n(G_s)$$

are martingales in the filtration $\{\mathcal{F}_s : s \in \mathbb{N}\}$ generated by the sequence. where $\mathcal{F}_s = \sigma(G_1, \ldots, G_s)$.

- Show that

$$Z_t = \exp\left(\theta \sum_{s=1}^{t} G_s - \frac{1}{2}\theta^2 t\right)$$

is a $\{\mathcal{F}_s :\}$-martingale.