Malliavin calculus, exercises 2, 03.02.2011

1. Let $W = (W_1, \dots, W_n)$ a random vector with independent standard gaussian coordinates $X_i \sim \mathcal{N}(0,1)$, and A an $n \times m$ matrix.

   Let $X = WA$. Show that X is gaussian. Compute the covariance and the characteristic function

   $$\varphi_X(t) = E_P\Big(\exp\big(\sqrt{(-1)}t \cdot X\big)\Big)$$

   where $t \in \mathbb{R}^m$ and $t \cdot x = \sum_{i=1}^m t_i x_i$.

2. Let (X, Y) jointly gaussian random variables, with $E(X) = E(Y) = 0$. Let $Y'(\omega) = \text{sign}(X(\omega))\, Y(\omega)$, where $\text{sign}(x) = \mathbf{1}(x > 0) - \mathbf{1}(x < 0)$.

   Show that Y' is gaussian but (X, Y') are not jointly gaussian.

3. Show that

   $$z(\mu, \sigma^2) := \int_{-\infty}^{\infty} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)dx = \sqrt{2\pi\sigma^2},$$

   for $\mu, \sigma \in \mathbb{R}$. Hint: to compute $z(0,1)^2$ one can write it as an integral in $\mathbb{R}^2$ and use polar coordinates.

4. Consider the one-dimensional gaussian random variable X with density

   $$p(x) = p(x; \mu, \sigma) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)$$

   Find a value $\lambda > 0$ such that

   $$E_P(\exp(\lambda X^2)) < \infty$$

   Show that for all λ, the moment generating function is finite:

   $$\varphi(\theta) := E_P(\exp(\theta X)) < \infty$$

   and compute its value.

5. Let f a function in $L^2(\mathbb{R}, \gamma)$, where $\gamma(x)$ is standard gaussian density. Assume that $x \mapsto f(x)$ is continuous has classical derivative $f'(x)$ at all $x \in \mathbb{R} \setminus C$, where the exceptional set is finite. Define $g(x) = f'(x)$ if $x \notin C$ and $g(x) \equiv 0$ when $x \in C$. Show that when $g \in L^2(\mathbb{R}, \gamma)$ g is the derivative in Sobolev sense of $f \in L^2(\mathbb{R}, \gamma)$.

   Hint: use integration by parts.

   Assume now that $x \to f(x)$ is differentiable at all $x \neq 0$, with $g = f' \in L^2(\mathbb{R}, \gamma)$ (where we define arbitrarily $g(0) = 0$) but f is discontinuous at 0:

   $$f(0-) = \lim_{x \uparrow 0} f(x) \neq f(0+) = \lim_{x \uparrow 0} f(x)$$

Show that $f \notin W^{1,2}(\gamma)$ that is f does not have a Sobolev derivative.

Hint: show this for the indicator $f(x) = \mathbf{1}(x \geq 0)$, and see what goes wrong when you integrate by parts. In this case the derivative exists only as a generalized a Dirac mass which is not in L^2.

6. Show that if $A \in \mathcal{B}(\mathbb{R}^n)$, (a Borel set) then the indicator $\mathbf{1}_A$ is not in the weighted Sobolev space $W^{1,2}(\mathbb{R}^n, \gamma^{\otimes n})$.

   Hint: $f \in W^{1,2}(\mathbb{R}^n, \gamma^{\otimes n})$ if and only if the random variable $f(\Delta W_1, \ldots, \Delta W_n)$ is Malliavin differentiable, where ΔW_i are independent standard gaussian random variables. You can assume first that A is a rectangle.

   Use the identity $\mathbf{1}_A(x) = \big(\mathbf{1}_A(x)\big)^2$ and assume that the Malliavin derivative $D\mathbf{1}_A$ which is the same as the Sobolev derivative exists, to find a contradiction.