Malliavin calculus, exercises 1, 26.01.2011

1. Check Abel summation formula: for $(x_0, x_1, \dots x_n), (y_0, y_n, \dots, y_n) \in \mathbb{R}^{n+1}$, Denoting $\Delta x_i = (x_i - x_{i-1})$, $\Delta y_i = (y_i - y_{i-1})$, check Abel summation formula

$$x_n y_n - x_0 y_0 = \sum_{i=1}^n x_{i-1}\Delta y_i + \sum_{i=1}^n y_{i-1}\Delta x_i + \sum_{i=1}^n \Delta y_i \Delta x_i$$

2.
   - Use Fubini theorem to prove the integration by parts formula

$$F(b)G(b) = F(a)G(a) + \int_a^b G(x)F(dx) + \int_a^b F(x)G(dx)$$

   when F and G have finite variation, and have no common jumps $(F(x+) - F(x-))(G(x+) - G(x-)) = 0 \ \forall x$
   (you can assume that F and G are continuous).
   - What is the formula when there are common jumps ? (look at Abel discrete formula)

3. Let G a standard gaussian random variable $E(G) = 0, E(G^2) = 1$.
   - Use integration by parts and induction to compute the moments $E(G^n)$ $n \in \mathbb{N}$.
   - Show that G and $(G^2 - 1)$ are uncorrelated.
   - Let X be $\mathcal{G}$-measurable and $Y \overset{P}{\perp\!\!\!\perp} \mathcal{G}$. Show that

$$E_P(f(X,Y)|\sigma(G))(\omega) = E_P(f(x,Y)\Big)\Big|_{x=X(\omega)}$$

   - Check this property of the conditional expectation using Kolmogorov definition:
   If the σ-algebra $\mathcal{H}$ is P-independent w.r.t the σ-algebra $\sigma(X) \vee \mathcal{G}$, then

$$E_P(X|\sigma(G) \vee \sigma(H))(\omega) = E_P(X|\sigma(G))(\omega)$$

   - Consider the operator ∂^* where $\partial^* f(x) = f(x)x - f'(x)$.
   Denote $h_0(x) \equiv 1$,
   $h_n(x) = \partial^* h_{n-1}(x) = (\partial^*)^n h_0(x) = (\partial^*)^n 1$.
   Use integration by parts formula $E(\partial f(G)h(G)) = E(f(G)\partial^* h(G))$
   Show that $E(h_n(G)) = 0$ and $E(h_n(G)h_m(G)) = 0$ when $n \neq m$.
   $h_n(x)$ are (unnormalized) Hermite polynomials.

4. Consider random variables $X(\omega), Y(\omega), Z(\omega) \in L^2(\Omega, \mathcal{F}, P)$. Let

$$H = \Big\{a(Y) + b(Y)Z : a, b \text{ Borel measurable functions}\Big\} \cap L_2(\Omega, \sigma(Y,Z), P)$$

   the subspace random variables in $L^2(\Omega, \sigma(Z,Y), P)$ which depend linearly on Z.

- Show that

$$\hat{X}(\omega) = E_P(X|\sigma(Y)) + \Big(Z(\omega) - E_P(Z|\sigma(Y))(\omega)\Big)\frac{\mathrm{Cov}_P(X, Z|\sigma(Y))(\omega)}{\mathrm{Var}_P(Z|\sigma(Y))(\omega)}$$

  is the orthogonal projection of X to the subspace H.
- Compute the conditional square error $E_P((\hat{X} - X)^2|\sigma(Y))(\omega)$.
- Compute the square error $E_P((\hat{X} - X)^2)$.

5. Let (X, Y) jointly gaussian, $E(X) = m_X$, $E(Y) = m_Y$ and covariance matrix

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_X & \Sigma_{YY} \end{pmatrix}$$

   - Use Bayes formula to compute the conditional distribution of X conditionally on $\sigma(Y)$.
   - Show that the conditional expectation has the form

$$E(X|\sigma(Y))(\omega) = \hat{b} + \hat{a}Y(\omega)$$