with $A' \subset A$ and $C \subset C'$, then

(5.4) $$p\text{-cap}(A', C') \geq p\text{-cap}(A, C).$$

The next theorem is due to F.W. Gehring G2, W.P. Ziemer Z2

5.5. <u>Thm</u>. Let A be bounded open set in $\mathbb{R}^n$ and $E = (A, C)$ a condenser. Then $\forall\, p > 1$

$$p\text{-cap}\, E = M_p(\Delta(C, \partial A; A)).$$

If $p = n$ this equality also holds if A is unbounded.

We say that (A, C) is a condenser in $\overline{\mathbb{R}}^n$ if there exists a q-isometry h such that (hA, hC) is a condenser in $\overline{\mathbb{R}}^n$. We define

$$n\text{-cap}(A, C) = n\text{-cap}(hA, hC).$$

It is clear that this definition does not depend on h.

Usually $p = n$ and we write $n\text{-cap}(A, C) = \text{cap}(A, C)$. By 5.5 many properties of the curve families can be directly generalized to capacities.

<u>Sets of capacity zero</u>. We now show that the condition $\text{cap}(A, C) = 0$ is independent of the open set A, $\overline{A} \neq \overline{\mathbb{R}}^n$. For this purpose we need the following "Poincaré inequality" (R12, p. 60, Lemma 3.3).

5.6. <u>Lemma</u>. Let $u \in C^\infty(\mathbb{R}^n)$ s.t. $u(x) = 0$ for $|x| \geq r > 0$. Then

$$\int_{\mathbb{R}^n} |u|^n \, dm \leq (2r)^n \int_{\mathbb{R}^n} |\nabla u|^n \, dm.$$

We use Poincaré's inequality to prove the following lemma.

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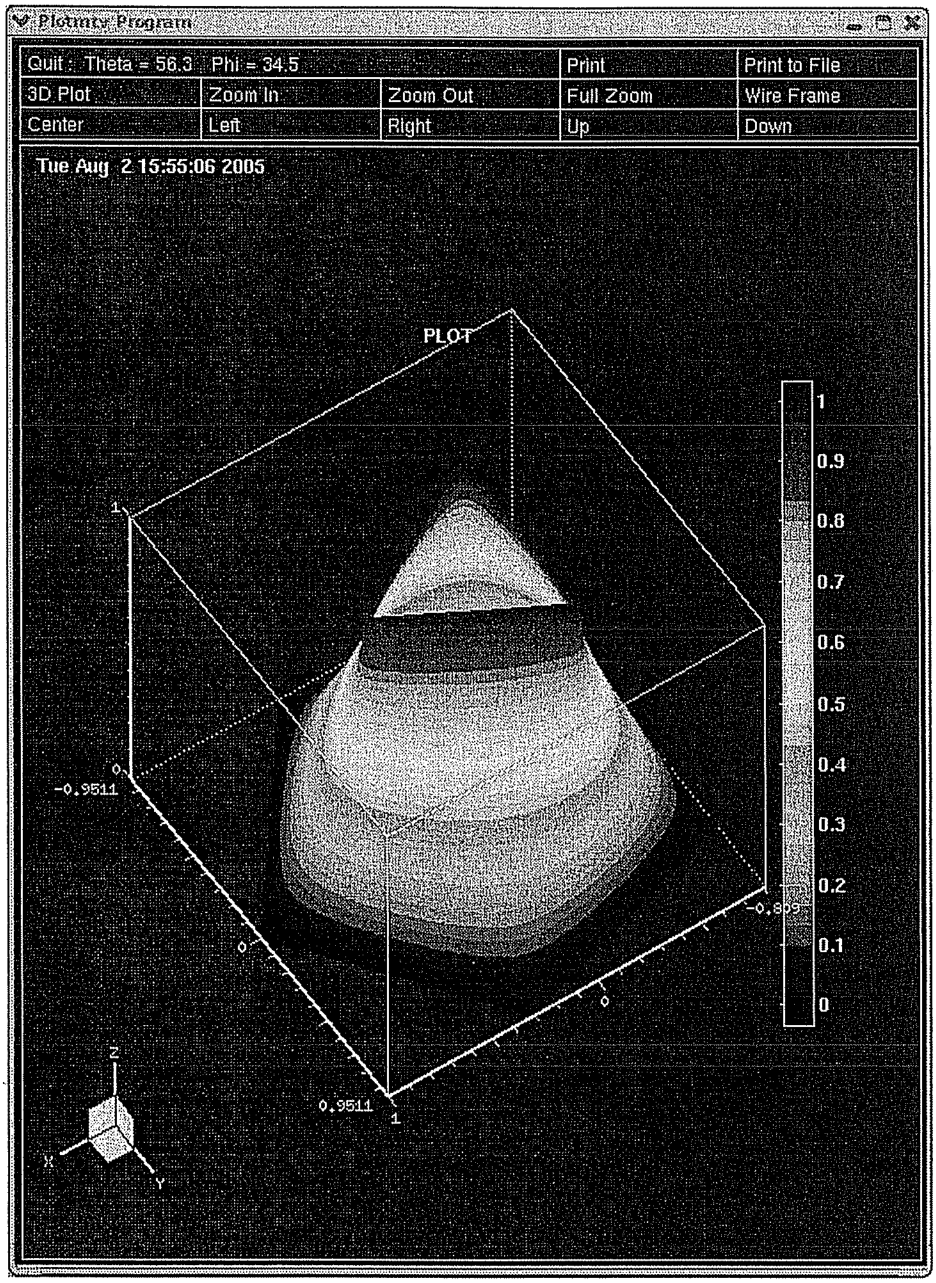
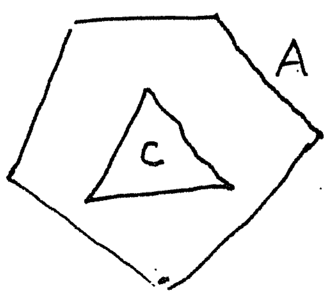


The infimum in (4.3) is a minimum under certain conditions. This minimizing function is shown in the above picture for the case indicated below:



5.7. Lemma. Let $E \subset B^n(R)$ be compact, $s>0$, and $E(t) = E + B^n(t)$, $t>0$ $(E(t) = \{z \in \mathbb{R}^n : |z-y|<t$ for some $y \in E\})$. Then for every $t>0$

$$M(\Delta(\partial E(t), E)) \le a(t)\, M(\Delta(\partial E(s), E))$$

where $a(t) \le a_1 t^{-n}$ for $t \in (0,s)$ and $a_1 \in (0,\infty)$ only depends on (n,s,R).

Proof. Fix first $t \in (0,s)$ and $\varepsilon>0$. Approximation property of ACL^p functions $\Rightarrow \exists\, u \in C_0^\infty(E(s))$ s.t. $u(x) \ge 1\ \forall x \in E$, $u(x)=0\ \forall x \notin E(s)$:

$$\int_{\mathbb{R}^n} |\nabla u|^n \, dm \le \varepsilon + M(\Delta(\partial E(s), E)) \overset{T.5.5}{=} \varepsilon + \operatorname{cap}(E(s), E)$$

There exists a constant $b_1 = b_1(n)$ and for every $t \in (0,s]$ $\varphi_t \in C_0^\infty(E(s))$ s.t. $0 \le \varphi_t(x) \le 1\ \forall x \in E(s)$ (ST, p. 171) and

(a) $\varphi_t(x) = 1 \quad \forall x \in E$

(b) $\varphi_t(x) = 0 \quad \forall x \in E(s) \setminus E(t)$

(c) $|\nabla \varphi_t(x)| \le b_1/t$

The function $v(x) = u(x)\varphi_t(x)$ is admissible in the sense of the def. (5.3) for $\operatorname{cap}(E(t), E)$, and therefore

$$M(\Delta(\partial E(t), E)) = \operatorname{cap}(E(t), E) \le \int_{\mathbb{R}^n} |\nabla v|^n \, dm$$

Because $(a+b)^n \le 2^{n-1}(a^n + b^n)$ $(a,b>0,\ n \ge 1)$ and further $|\nabla v(x)|^n \le 2^{n-1}(|\nabla u(x)|^n \varphi_t(x)^n + |\nabla \varphi_t(x)|^n |u(x)|^n)$ we obtain by (a) and (c)

$$(*) \quad \operatorname{cap}(E(t), E) \le \int |\nabla v|^n \, dm \le 2^{n-1} \int_{\mathbb{R}^n} |\nabla u|^n \, dm + (2b_1)^{n-1} t^{-n} \int_{\mathbb{R}^n} |u(x)|^n \, dm$$

Now Poincaré inequality Lemma 5.6 yields

$$(**) \qquad \int_{R^n} |u(x)|^n \, dm \le 2^n (R+s)^n \int_{R^n} |\nabla u(x)|^n \, dm$$

and therefore $(*)$ and $(**)$ imply for $t \in (0,s)$

$$\operatorname{cap}(E(t), E) \le a(t) \int |\nabla u|^n dm \le a(t)(\varepsilon + \operatorname{cap}(E(t), E))$$

$$a(t) = 2^{n-1}(1 + 2^n b_1^n (R+s)^n / t^n).$$

Letting $\varepsilon \to 0$ we get the assertion for the case $t \in (0,s)$.

In the case $t \ge s$ we have

$M(\Delta(\partial E(t), E)) \le M(\Delta(\partial E(s), E)) \le \max\{1, a(s)\} M(\Delta(\partial E(s), E))$

20.10.

so that we set $a(t) = \max\{1, a(s)\}$ for $t \ge s$.

5.8. <u>Rmk.</u> Set $Q = [0,1]^{n-1} \times \{0\}$, $Q_j = Q + j \cdot 2^{-k} e_n$, $j = 1, \ldots 2^k$, $E = \cup_{j=1}^{2^k} Q_j$, $t = \frac{1}{3} 2^{-k}$. Then $E(t)$ has 2^k components $E(t)_j$, $j = 1, \ldots, 2^k$ and

$$\operatorname{cap}(E(t), E) \ge \sum_{j=1}^{2^k} \operatorname{cap}(E(t)_j, E) \ge 2^k \left(\tfrac{1}{3} 2^{-k}\right)^{1-n} = \tfrac{1}{3} t^{-n}$$

Thus the estimate $a(t) \le a_1 t^{-n}$ of Lemma 5.7 is of correct form.

5.9. <u>Def.</u> A set $E \subset \mathbb{R}^n$ is said to be of <u>capacity zero</u> if (a) E is compact (b) $\exists$ bounded open $A \subset \mathbb{R}^n$ s.t. $E \subset A$ and $\operatorname{cap}(A, E) = 0$. Otherwise we say that E is of <u>positive capacity</u>. These are denoted $\operatorname{cap} E = 0$ and $\operatorname{cap} E > 0$.

A compact set $E \subset \overline{\mathbb{R}}^n$ is of capacity zero if E may be mapped by a Möbius transformation h s.t. $hE \subset B^n$ and $\operatorname{cap} h(E) = 0$ in the above sense.

5.10. <u>Rmk.</u> We now show that this definition is independent of the open bounded set A. Let $E \neq \emptyset$ be compact, $E \subset A_j$, $j=1,2$, $A_j \subset B^n(R)$, $j=1,2$. Then $A_j \subset E + B^n(2R)$, $j=1,2$ so that 5.7 $\Rightarrow$

$$\operatorname{cap}(A_j, E) \le \operatorname{cap}(E + B^n(t_j), E) \le$$

$$a(t_j)\operatorname{cap}(E+B^n(2R), E) \le a(t_j)\operatorname{cap}(A_j, E), \quad j=1,2$$

where $t_j = d(E, \partial A_j)$. Writing $D = E + B^n(2R)$ we have

$$\operatorname{cap}(A_1, E) \le a(t_1)\operatorname{cap}(D, E) \le a(t_1)\operatorname{cap}(A_2, E)$$
$$\le a(t_1)a(t_2)\operatorname{cap}(D, E) \le a(t_1)a(t_2)\operatorname{cap}(A_1, E)$$

In particular: $\operatorname{cap}(A_1, E) = 0 \Leftrightarrow \operatorname{cap}(A_2, E) = 0$

5.11. <u>Rmk.</u> (1) If $F \subset \mathbb{R}^n$ is a compact set, $\operatorname{cap} F = 0$, then $\Lambda_\alpha(F) = 0$ $\forall \alpha > 0$ where Λ_α is α-dimensional Hausdorff measure. Therefore H-dim $F = 0$. In particular: $\operatorname{int} F = \emptyset$ and F is totally disconnected.

(2) For $n=2$ one sometimes uses the logarithmic capacity. H. Wallin W1 has proved that for $n=2$: $\log\text{-}\operatorname{cap} E = 0 \Leftrightarrow 2\text{-}\operatorname{cap} E = 0$, $E \subset \mathbb{R}^2$. In W2 an example of a set $E \subset \mathbb{R}^n$ with H-dim $E = 0$ and $n\text{-}\operatorname{cap} E > 0$ is given. See also MK.

(3) In the case $\operatorname{cap} E > 0$ there is no such a notion as the magnitude of the capacity as a real number (in the case $\operatorname{cap} E = 0$, we may think that 0 is the magnitude) because the above def. depends on the condenser (A, E). The set function $c(E)$ is for the case $\operatorname{cap} E > 0$ a quantitative "magnitude" for the size of E.

(4) For numerical computation of capacity, see Rasila's talk on http://users.utu.fi/ripekl/mad07

5.11.b. Lemma Let $G \subset \mathbb{R}^n$ be a domain, let $u: G \to \mathbb{R}$ be an ACL^p-function, $-\infty < a < b < \infty$ and let $A, B \subset G$ be nonempty sets such that $u(x) \le a$ for all $x \in A$ and $u(x) \ge b$ for all $x \in B$. Then

$$M_p(\Delta(A,B;G)) \le (b-a)^{-p} \int_G |\nabla u|^p \, dm .$$

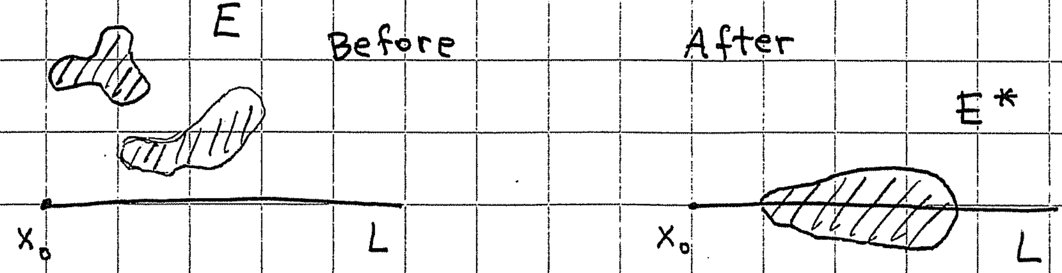
Proof. Here an outline is given of the proof in CGQM, Le 7.6. Let $v(x) = (u(x)-a)/(b-a)$. Then $v(x) \ge 1$ $\forall x \in B$ and $v(x) \le 0$ $\forall x \in A$. Fix a rectifiable path $\gamma \in \Delta(A,B;G)$ and let $\gamma^\circ : [0,c] \to G$, $c = \ell(\gamma) < \infty$ be its normal representation. Then

$$1 \le v(\gamma^\circ(c)) - v(\gamma^\circ(0)) \le \int_0^c |(v \circ \gamma^\circ)'(t)| \, dt$$

$$= \int_0^c |\nabla v(\gamma^\circ(t))| \, |\gamma^{\circ\prime}(t)| \, dt = \int_\gamma |\nabla v| \, dt$$

and $$M_p(\Delta(A,B;G)) \le \int_G |\nabla v|^p dm = (b-a)^{-p} \int_G |\nabla u|^p dm . \quad \square$$

5.12. Spherical symmetrization If $x_0 \in \mathbb{R}^n$, $E \subset \overline{\mathbb{R}}^n$ and L is a ray from the point x_0 to ∞, $L = \{t x_0 ; t \ge 1\}$, then the spherical symmetrization E^* of E in L is defined as follows: (1) $x_0 \in E^*$ iff $x_0 \in E$ (2) $\infty \in E^* \Leftrightarrow \infty \in E$
(3) For $r \in (0,\infty)$, $E^* \cap S^{n-1}(x_0,r) \ne \emptyset$ iff $E \cap S^{n-1}(x_0,r) \ne \emptyset$ and in this case $E^* \cap S^{n-1}(x_0,r)$ is a closed spherical cap with center at L s.t. $m_{n-1}(E^* \cap S^{n-1}(x_0,r)) = m_{n-1}(E \cap S^{n-1}(x_0,r))$.

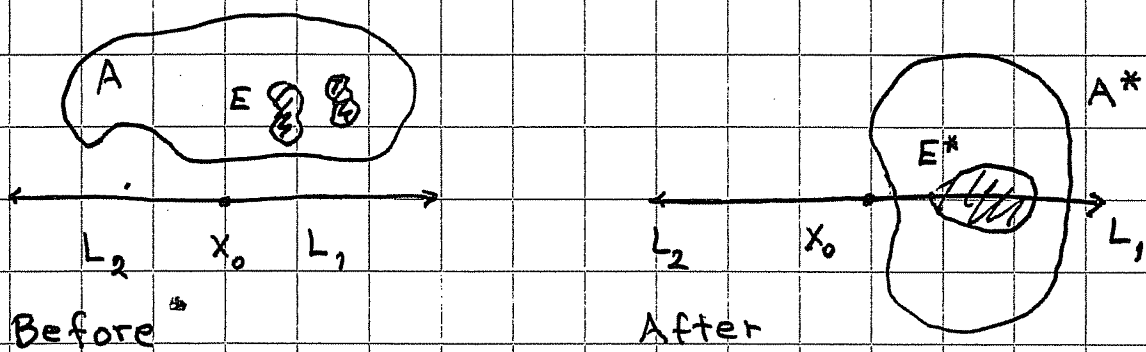


Below we will apply symm. only to sets that are either open or closed.

Let (A, C) be a condenser in $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Let L_1 and L_2 be two opposite rays starting from x_0 and let C^* be the spherical symm. of C in L_1 and B the symm. of $\mathbb{R}^n \setminus A$ in L_2. Write $A^* = \mathbb{R}^n \setminus B$. It can be shown that (A^*, C^*) is a condenser. A central result is

5.13. Lemma. If (A, C) is a condenser in $\mathbb{R}^n$, then

$$p\text{-cap}(A, C) \geq p\text{-cap}(A^*, C^*); \quad p > 1.$$



The equality holds in Lemma 4.13 if $(A, C) = (A^*, C^*)$ (e.g. $x_0 = 0$, $C = [0, e_1]$, $A = B^2(2)$, $L_1 =$ the posit. x_1-axis). It should be noticed that $p\text{-cap}(A^*, C^*)$ depends essentially on the choice of x_0. It is possible that $p\text{-cap}(A^*, C^*) = 0 < p\text{-cap}(A, C)$.

5.14. The modulus and capacity of a ring. A domain D in $\overline{\mathbb{R}}^n$ is termed a ring, if $\overline{\mathbb{R}}^n \setminus D$ has exactly two components. If they are C_0 and C_1 we write $D = R(C_0, C_1)$. The modulus and capacity of the ring D are defined by

$$\text{(5.15)} \quad \begin{cases} \operatorname{cap} R(C_0, C_1) = M(\Delta(C_0, C_1)), \\ \operatorname{mod} R(C_0, C_1) = \left((\operatorname{cap} R(C_0, C_1)) / \omega_{n-1} \right)^{1/(1-n)}. \end{cases}$$

where ω_{n-1} is the $(n-1)$-dim. measure of S^{n-1}. If $C_0 = \overline{B}^n$ and $C_1 = \overline{\mathbb{R}}^n \setminus B^n(a)$, $a > 1$, then $\operatorname{mod} R(C_0, C_1) = \log a$.

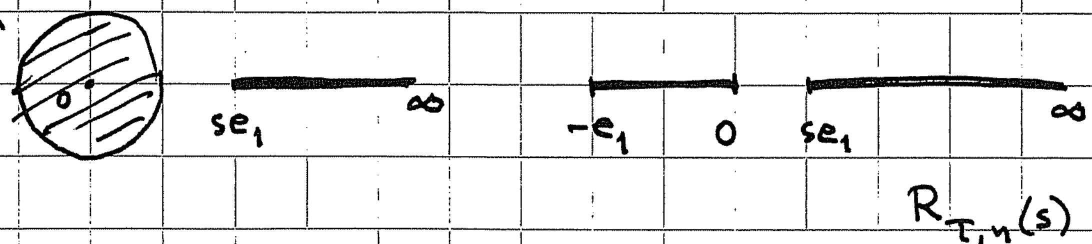
Clearly, a ring is a special case of a condenser. Below we always assume that C_0 is bdd and we identify the ring $R(C_0, C_1)$ and the condenser $(\mathbb{R}^n \setminus C_1, C_0)$. For $n = 2$ $\operatorname{mod} R = t$ iff the ring R may be conformally mapped onto the annulus $\{z \in \mathbb{R}^2 : 1 < |z| < e^t\}$.

5.16. <u>The rings of Grötzsch and Teichmüller</u> Let

$$R_{G,n}(s) = (\mathbb{R}^n \setminus \{te_1 : t \geq s\}, \overline{B}^n), \quad s \in (1, \infty)$$
$$R_{T,n}(s) = (\mathbb{R}^n \setminus \{te_1 : t \geq s\}, [-e_1, 0]), \quad s \in (0, \infty)$$

We introduce abbreviations

$$\text{(5.17)} \quad \begin{cases} \gamma_n(s) = \operatorname{cap} R_{G,n}(s) = \gamma(s) \\ \tau_n(s) = \operatorname{cap} R_{T,n}(s) = \tau(s) \end{cases}$$



Convention: $\gamma_n(s) = \infty \ \forall s \leq 1$, $\gamma_n(\infty) = 0$
$\tau_n(t) = \infty \ \forall t \leq 0$, $\tau_n(\infty) = 0$

The functions $\Phi(s)$, Ψ are defined by

$$\text{(5.18)} \quad \begin{cases} \operatorname{cap} R_{G,n}(s) = \omega_{n-1} (\log \Phi(s))^{1-n} \\ \operatorname{cap} R_{T,n}(s) = \omega_{n-1} (\log \Psi(s))^{1-n} \end{cases}$$

which implies $\operatorname{mod} R_{G,n}(s)=\log \Phi(s)$, $\operatorname{mod} R_{T,n}(s)=\log\Psi(s)$.

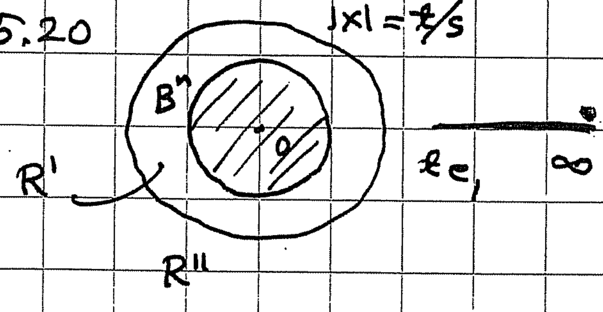
5.19. Lemma. For $s>0$ and $1<r<1+s$
$$\tau(s)^{1/(1-n)} \ge \gamma(r)^{1/(1-n)} + \gamma((s+1)/r)^{1/(1-n)} .$$

Proof. Let $\Gamma_1=\Delta([-e_1,0], S^{n-1}(-e_1,r); B^n(-e_1,r))$,
$\Gamma_2=\Delta(S^{n-1}(-e_1,r),[se_1,\infty); \mathbb{R}^n\setminus B^n(-e_1,r))$,
$\Gamma=\Delta([-e_1,0],[se_1,\infty))$.

3.27 $\Rightarrow$
$$M(\Gamma)^{1/(1-n)}=\tau(s)^{1/(1-n)}\ge M(\Gamma_1)^{1/(1-n)}+M(\Gamma_2)^{1/(1-n)}=\gamma(r)^{1/(1-n)}+\gamma((s+1)/r)^{1/(1-n)}$$

5.20. Cor. $\gamma(s)=2^{n-1}\tau(s^2-1)$ (i.e. = holds in 5.19 for $r=\sqrt{1+s}$).
Proof. Set $r=\sqrt{1+s}$ in 5.19. The proof follows from 3.29.

5.21. Lemma. $\Phi(t)/t$ is increasing and $\Psi(t-1)=\Phi(\sqrt{t})^2$, $t>1$.
Proof. Let $1<s<t$, $R=R_{G,n}(t)$ and let R', R'' be the two rings into which the sphere $|x|=t/s$ splits R. 3.12 & 3.27 $\Rightarrow$
$$\log\Phi(t)=\operatorname{mod}R\ge\operatorname{mod}R'+\operatorname{mod}R''=\log\frac{t}{s}+\log\Phi(s)$$
$\Rightarrow \Phi(t)/t \ge \Phi(s)/s$.
The functional identity follows from 5.20



Note that 5.21 implies that
$\Phi(s)$ is strictly increasing $\Rightarrow$
$\gamma(s)$ is str. decreasing.

We define the Grötzsch constant λ_n by

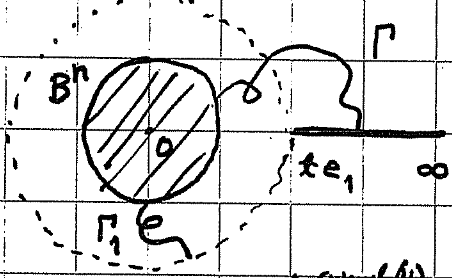
(5.22) $\qquad \log\lambda_n=\lim_{t\to\infty}(\log\Phi(t)-\log t)$ $\quad [5.21\Rightarrow \exists \lim\in\dot{\mathbb{R}}^1]$ $\quad \dot{\mathbb{R}}^1 = \mathbb{R}\cup\{\infty\}$

For $n=2$: $\lambda_2=4$ and $\lambda_n\in[4,2e^{n-1})$, $n\ge3$.

5.23. Lemma The number λ_n satisfies $\lambda_n \in [4, 2e^{n-1})$ and

(1) $t \le \Phi(t) \le \lambda_n t, \quad t > 1,$

(2) $t+1 \le \Psi(t) \le \lambda_n^2 (t+1), \quad t > 0.$



Pf. The lower bound in (1) follows from

$\gamma(t) = M(\Gamma) \le M(\Gamma_1) = \omega_{n-1} (\log t)^{1-n}$

The upper bound in (1) follows from (5.22). The ineq. $\lambda_n \in [4, 2e^{n-1})$ is in [AN2]. (2) follows from 5.21 and (1)

Lemma 5.23 gives the foll. estimates for $s > 1$

$$(5.24)\quad \begin{cases} \omega_{n-1} (\log \lambda_n s)^{1-n} \le \gamma_n(s) \le \omega_{n-1} (\log s)^{1-n} \\ \omega_{n-1} (\log \lambda_n^2 s)^{1-n} \le \tau_n(s-1) \le \omega_{n-1} (\log s)^{1-n} \end{cases}$$

For $s > 1$ close to 1 we have better estimates by the foll. lemma.

5.25. Lemma For $s \in (1, \infty)$ and $n \ge 2$

(1) $$\gamma_n(s) \le \omega_{n-1} \mu(1/s)^{1-n}.$$

Equality holds in (1) for $n = 2$

(2) $2^{n-1} c_n \log \frac{s+1}{s-1} \le \gamma_n(s) \le 2^{n-1} c_n \mu\left(\frac{s-1}{s+1}\right) < 2^{n-1} c_n \log\left(4 \frac{s+1}{s-1}\right).$

Further, if $s \in (0, \infty)$ and $a = 1 + \frac{2}{s}(1 + \sqrt{1+s})$, then

(3) $$c_n \log a \le \tau_n(s) \le c_n \mu(1/a) < c_n \log(4a)$$

and $(1 + 1/\sqrt{s})^2 \le a \le (1 + 2/\sqrt{s})^2$ hold.

Pf. (1) See [AN1]. Recall the formula (3.23) for $n = 2$.

(2) The lower bound follows from the sph. cap ineq. see CGQM, 7.26

(3) Follows from 5.20 and (2).

5.26. Rmk. Because $\mu(r) > \log \frac{1}{r}$ by (3.26), 5.25 (1) gives a better upper bound than (5.24).

5.27. Summary. Write $u_1 = \omega_{n-1}\,\mu(1/s)^{1-n}$, $u_2 = 2^{n-1}c_n\,\mu\left(\frac{s-1}{s+1}\right)$
$v_1 = \omega_{n-1}(\log \lambda_n s)^{1-n}$, $v_2 = 2^{n-1}c_n \log\frac{s+1}{s-1}$. (5.24) & 5.25 $\Rightarrow$

$$(5.28)\qquad \max\{v_1, v_2\} \le \gamma_n(s) \le \min\{u_1, u_2\}.$$

See the graph on p. 92 of CGQM

5.29. Ex. Use the conformal invariance and an aux. map in GM to show that

$$M(\Delta([0, se_1], [te_1, e_1])) = \tau\left(\frac{t-s}{s(1-t)}\right) \qquad 0 < s < t < 1$$

$$M(\Delta(S^{n-1}, [se_1, te_1])) = \gamma\left(\frac{st-1}{t-s}\right), \quad 1 < s < t < \infty.$$

5.30. Hyperbolic metric and capacity. Let $J[x,y]$ be a geod. segment of the hyp. metric, $x, y \in B^n$ and $T_x \in \mathcal{M}(B^n)$. (2.18) $\Rightarrow$

$$(5.31)\quad \begin{cases} \operatorname{cap}(B^n, J[x,y]) = \operatorname{cap}(B^n, T_x J[x,y]) = \\ \operatorname{cap}(B^n, [0, |T_x y| e_1]) = \gamma\left(1/\operatorname{th}\frac{\rho(x,y)}{2}\right) \end{cases}$$

Substitution into 5.25(2) yields

$$(5.32)\quad \begin{cases} 2^{n-1}c_n\,\rho(x,y) \le \operatorname{cap}(B^n, J[x,y]) \le 2^{n-1}c_n\,\mu\left(\dfrac{1-\operatorname{th}\frac{\rho(x,y)}{2}}{1+\operatorname{th}\frac{\rho(x,y)}{2}}\right) \\ = 2^{n-1}c_n\,\mu\Bigg(\underbrace{\dfrac{\operatorname{ch}\frac{\rho}{2}-\operatorname{sh}\frac{\rho}{2}}{\operatorname{ch}\frac{\rho}{2}+\operatorname{sh}\frac{\rho}{2}}}_{=e^{-\rho}}\Bigg) \le 2^{n-1}c_n\,\mu(e^{-\rho}) \le 2^{n-1}c_n \log\left(4e^{\rho(x,y)}\right) \\ \qquad = 2^{n-1}c_n\,(\rho(x,y) + \log 4) \end{cases}$$

where the ineq. $\mu(t) < \log\frac{4}{t}$, see (3.26) was also used. For large values of $\rho(x,y)$, (5.32) is quite accurate. When $\rho(x,y)$ is small, better bounds follow from 5.25(1) and (5.24)

5.33. Lemma Let $x, y \in B^n$ and E a continuum with $x, y \in E$. Then

$$\operatorname{cap}(B^n, E) \ge \operatorname{cap}(B^n, J[x,y]) = \gamma\left(1/\operatorname{th}\frac{\rho(x,y)}{2}\right).$$

Proof. As in (5.31) we get

$$\mathrm{cap}(B^n, E) = \mathrm{cap}(B^n, T_x E) \geq \mathrm{cap}(B^n, (T_x E)^*)$$

where Lemma 5.13 was applied and $*$ stands for the spher. symm in x_1-axis. (2.18) $\Rightarrow [0, (\mathrm{th}\frac{\rho(x,y)}{2})e_1] \subset (T_x E)^*$. The claim follows from (5.31).

5.34. Functional inequalities for $\tau_n(t)$. For $n=2$ we have

$$\tau_2(t) = 2^{1-2}\gamma_2(\sqrt{1+t}) = 2^{-1}2\pi/\mu(1/\sqrt{1+t}) = \pi/\mu(1/\sqrt{1+t})$$

The properties of the μ function imply that

$$\tau_2(t) = 2\tau_2\left(4(t+\sqrt{t(1+t)})(1+t+\sqrt{t(1+t)})\right)$$

hence $$\tau_2(1) = 2\tau_2\left(4(1+\sqrt{2})(2+\sqrt{2})\right) = 2\tau_2\big(\underbrace{4\sqrt{2}(1+\sqrt{2})^2}_{\sim 32.9}\big)$$

For $n \geq 3$ no such identities are known.

We now prove functional inequalities for $\tau_n(t)$, $n \geq 2$.

5.35. Lemma. For $s>0$ we have $(\tau = \tau_n, n \geq 2)$

(1) $\tau(s) \leq \gamma(1+2s) = 2^{n-1}\tau(4s^2+4s)$,

(2) $\tau(s) \leq 2\tau(2s+2s\sqrt{1+1/s})$,

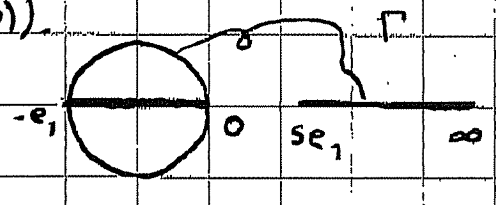
(3) $\tau(s) \leq \tau(t) + \tau\left(\frac{s(1+t)}{t-s}\right)$, $0<s<t<\infty$,

(4) $\tau(u) \leq \tau\left(\frac{uv}{u+v+1}\right) \leq \tau(u)+\tau(v)$, $u, v > 0$.

Pf. (1) Let $\Gamma = \Delta(S^{n-1}(-e_1/2, 1/2), [se_1, \infty))$.

Then 5.20 $\Rightarrow$

$$M(\Gamma) = \gamma(1+2s) = 2^{n-1}\tau(4s^2+4s)$$

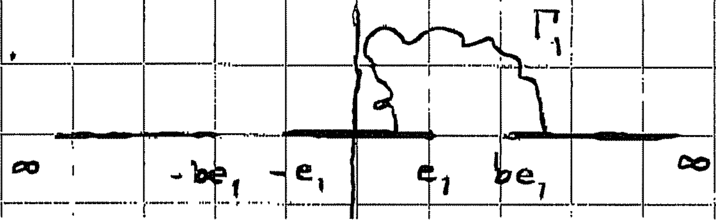


On the other hand $\tau(s) \leq M(\Gamma)$.

(2) We map $R_{T,n}(s)$ by a Möbius map to a ring with compl. components $[-e_1, e_1]$ and $\bar{R}^n \setminus (-be_1, be_1)$, $b = 1+2s(1+\sqrt{1+1/s})$.

Let $\Gamma_1 = \Delta([0, e_1], [be_1, \infty), \{x : x_1 > 0\})$. Then

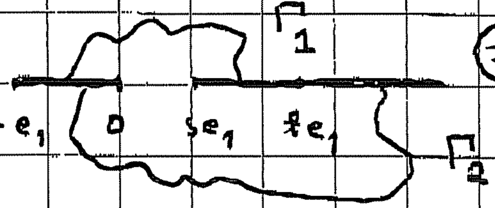
$$\tau(s) = 2M(\Gamma_1) \leq 2\tau(b-1).$$



(75)

(3) Let $\Gamma_1 = \Delta([-e_1, 0], [se_1, te_1])$

$\Gamma_2 = \Delta([-e_1, 0], [te_1, \infty))$



Then

$$\tau(s) \le M(\Gamma_1 \cup \Gamma_2) \le M(\Gamma_1) + M(\Gamma_2) = \tau\left(\frac{s(1+t)}{t-s}\right) + \tau(t)$$

(4) Write $u = t$, $v = \frac{s(1+t)}{t-s}$. Then $s = \frac{uv}{u+v+1}$ and the 2nd ineq. follows from (3). The 1st one follows since τ is decr.

5.36. Cor. $\tau(s) \le 2\tau(\sqrt{s}) \le 2^n \tau(s)$, $s > 0$.

Pf. The 1st ineq. follows from 5.35(2) since τ is decr.

The 2nd ineq. —"— 5.35(1).

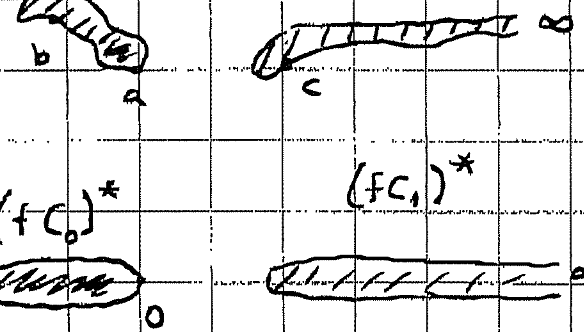
5.37. Lemma. Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbb{R}}^n$ and $a, b \in C_0$, $c, \infty \in C_1$ distinct points. Then

$$\operatorname{cap} R = M(\Delta(C_0, C_1)) \ge \tau\left(\frac{|a-c|}{|a-b|}\right).$$

Here = holds e.g. if $C_0 = [-e_1, 0]$, $a = 0$, $b = -e_1$, $C_1 = [se_1, \infty]$, $c = se_1$, $d = \infty$.

Pf. Note first that both the LHS and RHS are invariant under the similarity map $f(x) = (x-a)/|a-b|$. Then $|f(b)| = 1$, $f(a) = 0$, $|f(c)| = |c-a|/|a-b|$. We perform the spher. symm. of fR in the negat. x_1-axis. We see that

$$\operatorname{cap} R = \operatorname{cap} fR \ge \tau\left(\frac{|a-c|}{|a-b|}\right).$$



5.38. Cor. Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbb{R}}^n$ and $a, b \in C_0$, $c, d \in C_1$ distinct points. Then

$$\operatorname{cap} R \geq \tau(|a,b,c,d|).$$

Pf. Möbius inv. $\Rightarrow$ may assume $a = 0$, $b = e_1$, $d = \infty$, $|c| = |a,b,c,d|$. The pf follows from 5.37.

5.39. Cor. Let $R = R(C_0, C_1)$ be a ring in $\overline{\mathbb{R}}^n$ and $a, b \in C_0$, $c, d \in C_1$ be distinct points of $\mathbb{R}^n$. Then

$$\operatorname{cap} R \geq \tau\left(\frac{t-s}{s(1-t)}\right)$$

where

$$s = \frac{|a-b|}{|a-b|+|a-c|+|c-d|}, \quad t = \frac{|a-b|+|a-c|}{|a-b|+|a-c|+|c-d|}.$$

The equality holds e.g. if $C_0 = [0, se_1]$, $a = 0$, $b = se_1$, $C_1 = [te_1, e_1]$, $c = te_1$, $d = e_1$ and $0 < s < t < 1$.

Pf. $a, b, c, d \neq \infty \overset{5.38}{\Longrightarrow} \operatorname{cap} R \geq \tau\left(\frac{|a-c||b-d|}{|a-b||c-d|}\right) \geq \tau\left(\frac{t-s}{s(1-t)}\right)$

where the last ineq. follows from $|b-d| \leq |a-b| + |a-c| + |c-d|$.

5.40. Cor. If $R = R(C_0, C_1)$ is a ring, then

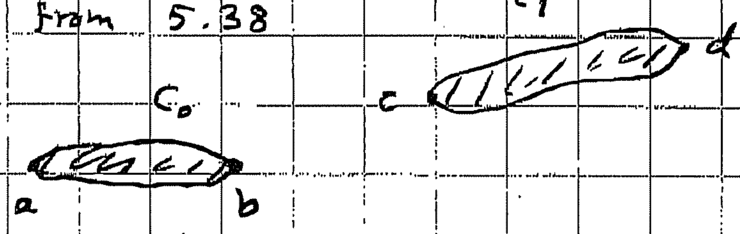
(1) $\operatorname{cap} R \geq \tau(1/(q(C_0)\,q(C_1)))$,

(2) $\operatorname{cap} R \geq \tau(4q(C_0, C_1)/(q(C_0)\,q(C_1)))$.

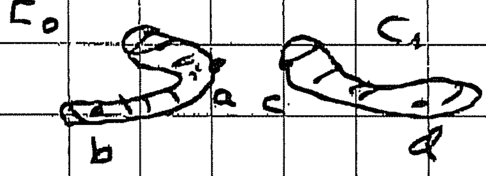
Pf. (1) Choose $a, b \in C_0$, $c, d \in C_1$ s.t. $q(a,b) = q(C_0)$ and $q(c,d) = q(C_1)$. Then

$$\frac{q(a,c)\,q(b,d)}{q(a,b)\,q(c,d)} \leq \frac{1}{q(C_0)\,q(C_1)}$$

Therefore (1) follows from 5.38



(2) Choose $a \in C_0$, $c \in C_1$ s.t. $q(a,c) = q(C_0, C_1)$ and $b \in C_0$ 77
$d \in C_1$ s.t. $q(a,b) \geq q(C_0)/2$, $q(c,d) \geq q(C_1)/2$. The pf foll. from 5.38



5.41. Lemma Let $E, F \subset \mathbb{R}^n$ be connected sets with $E \cap F = \emptyset$
$d(E) > 0, d(F) > 0$. Then

$$M(\Delta(E,F)) \geq \tau(4m^2 + 4m) \geq c_n \log(1 + 1/m)$$

where $m = d(E,F)/\min\{d(E), d(F)\}$ and c_n is as in

Pf. Fix $a \in E$, $c \in F$ s.t. $|a-c| = d(E,F)$ and $b \in E$, $d \in F$ s.t.
$|a-b| = d(E)/2$ and $|c-d| = d(F)/2$. 5.38 $\Rightarrow$

$$M(\Delta(E,F)) \geq \tau\left(\frac{|a-c||b-d|}{|a-b||c-d|}\right) \geq \tau\left(\frac{|a-c|(|b-a|+|a-c|+|c-d|)}{|a-b||c-d|}\right) = \tau(u)$$

where

$$u = \frac{2\,d(E,F)(d(E) + 2d(E,F) + d(F))}{d(E)\,d(F)} \leq 2m + 4m^2 + 2m$$

and hence $M(\Delta(E,F)) \geq \tau(4m^2 + 4m) \underset{5.25(3)}{\geq} c_n \log(1 + 1/m)$

5.42. Cor. Let $E, F \subset \mathbb{R}^n$ be connected with $E \cap F = \emptyset$ and
$0 < d(E) \leq d(F)$. Then

$$M(\Delta(E,F)) \geq 2^{1-n}\tau(d(E,F)/d(E))$$

Pf. 5.41 & 5.35(1) $\Rightarrow M(\Delta(E,F)) \geq \tau(4m^2 + 4m) \geq 2^{1-n}\tau(m)$.

We next generalize (5.32) by replacing the ring
$R(\mathbb{R}^n \setminus B^n, J[x,y])$ with a general ring $R(E,F)$ and $s(x,y)$
by $\min\{j_{\mathbb{R}^n \setminus E}(F), j_{\mathbb{R}^n \setminus F}(E)\}$. Recall that

$$j_D(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right); \quad j_D(A) = \sup\{j_D(x,y) : x, y \in A\}.$$

5.43. Lemma. Let $R = R(E,F)$ be a ring. If $\infty \notin E \cup F$ then

$$\operatorname{cap} R \geq c_n \min\{ j_{\mathbb{R}^n \setminus E}(F), j_{\mathbb{R}^n \setminus F}(E)\}$$

If $\infty \in F$, then

$$\operatorname{cap} R \geq c_n \, j_{\mathbb{R}^n \setminus F}(E).$$

Pf. Because $j_{\mathbb{R}^n \setminus E}(F) \leq \log\left(1 + \frac{d(E)}{d(E,F)}\right)$, the pf follows directly from 5.41 and 5.37.

Next we apply 5.43 with $F = \mathbb{R}^n \setminus B^n$, $E = J[x,y]$, $x, y \in B^n$ to obtain

$$M(\Delta(E,F)) \geq c_n j_{B^n}(J[x,y]) = c_n j_{B^n}(x,y) \geq \frac{c_n}{4} \varrho(x,y).$$

In conclusion, 5.43 yields (5.32) [however not with the same constant factor]. Therefore one may consider 5.43 as a generalization of (5.32).

5.44. Literature 5.12: G1, G6, 5.13: S1, 5.21: G2 5.23: G1, AN1, AN2, 5.25: AN1, VU10, 5.35: VU12, 5.37: G1, G6, 5.41 VU12

5.45. Question. I do not know whether the lower bound 5.42 is sharp. What is the largest constant in 5.42 (in place of 2^{1-n}) for which it is true?