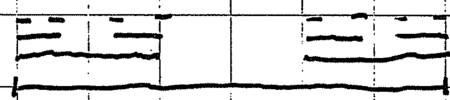
4. Excursion to real analysis

Here a few basic facts about absolutely continuous functions will be recalled. See HS, pp. 70, 283 - 285

4.1. Cantor's construction. From $I = [0,1]$ remove the middle third $(1/3, 2/3)$. Next from the remaining $[0, 1/3]$, $[2/3, 1]$ remove the middle third. After n steps of this construction we have 2^n closed intervals each of length 3^{-n}, let C_n be their union. The set $C = \bigcap_{n=1}^{\infty} C_n$ is the Cantor $1/3$-set. Moreover, $m(C) = 0$.

4.2. Thm. The set C is compact, contains no interior points, contains no isolated points.



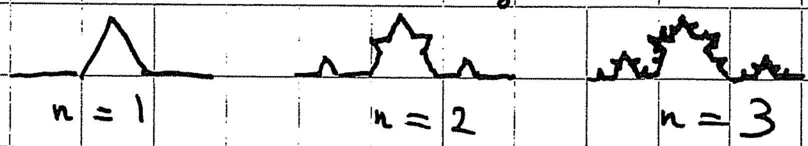
Cantor's construction can be extended to $\mathbb{R}^n$. There are also many modifications, e.g. one third could be replaced by some other fraction.

4.3 Lebesgue's singular function Define $\psi : [0,1] \to [0,1]$ using Cantor's construction as follows: $\psi(0) = 0$, $\psi(x) = (2k-1)/2^n$ for $x \in I_{n,k}$, $n = 1, 2, \ldots$; $k = 1, 2, \ldots, 2^{n-1}$, $\psi(x) = \sup\{\psi(t) : t \in [0,1] \setminus C,\ t < x\}$, $x \in C \cap (\mathbb{R} \setminus \{0\})$. Here $I_{n,k}$ are the "removed" intervals of the Cantor construction.

Then (1) ψ is defined for all $x \in [0,1]$ (2) ψ is monotone increasing (3) ψ is continuous (4) $\psi'(x) = 0$ $\forall x \in [0,1] \setminus C$ (5) $\psi[0,1] = [0,1]$.

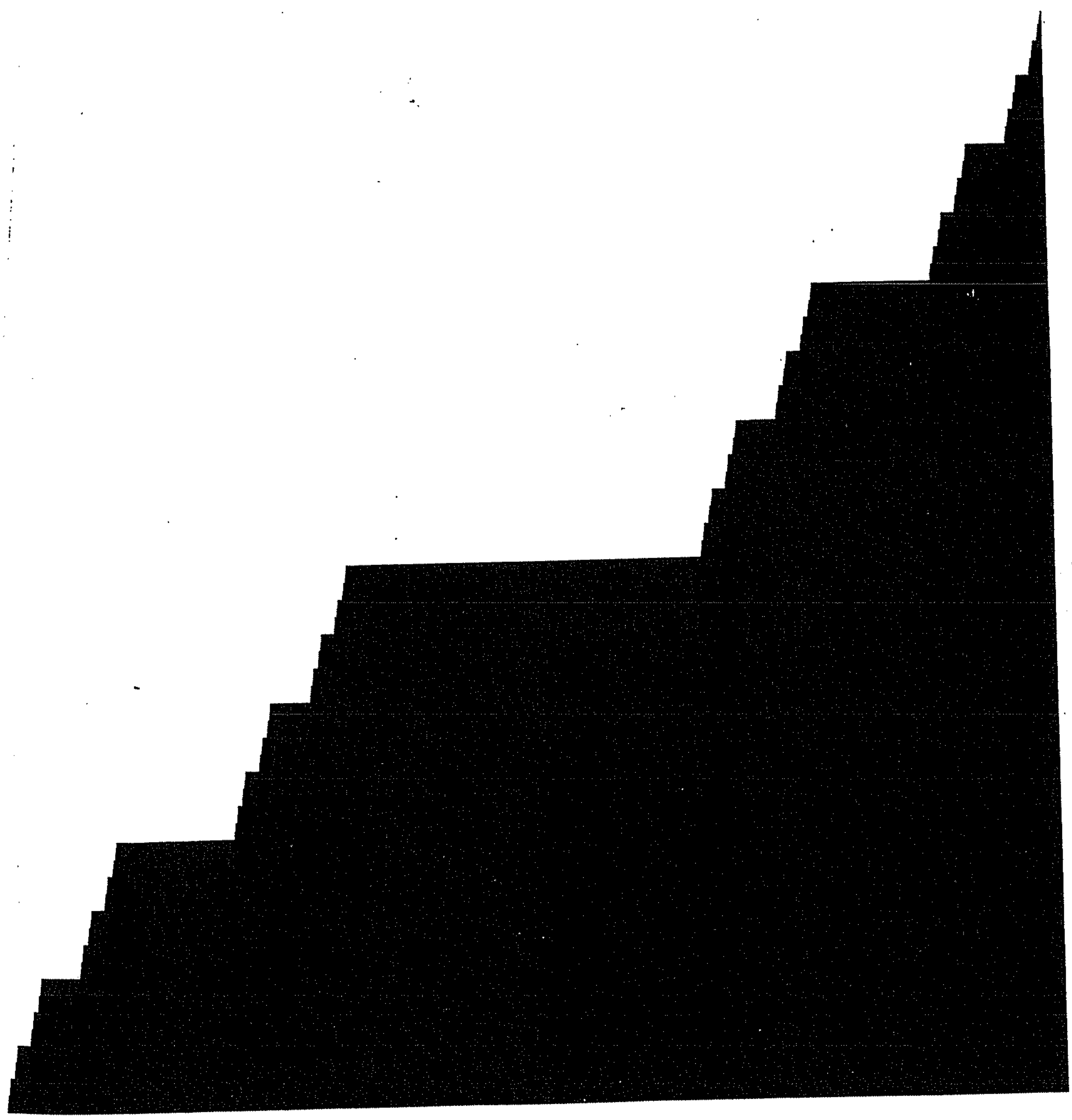
4.4. Rmk. The popular term "fractal" cannot be defined in a simple way. Usually this refers to sets constructed by a deterministic algorithm and these sets have "self-similar" geometric

geometric structure. The Cantor set is the simplest example of a fractal set. Recall also the snowflake curve constructed recursively:



This construction "converges" to a non-rectifiable curve, which is called the snowflake curve, also a fractal set.

4.5. Skyline of Lebesgue's singular function. The region below the graph of L's sing. function and above the x-axis looks as follows (this also suggests the term "Devil's staircase")



4.6. Def. A function $f:[a,b]\to\mathbb{R}$ is said to be absolutely continuous on $[a,b]$ if for every $\varepsilon>0$ there is a $\delta>0$ such that for every finite family of disjoint open subintervals $\{(c_k,d_k):k=1,\dots,n\}$ with

$$\sum_{k=1}^{n}(d_k-c_k)<\delta \implies \sum_{k=1}^{n}|f(d_k)-f(c_k)|<\varepsilon.$$

4.7. EX. (1) One can show that the indefinite integral of an L^1-function is abs. cont. (considered as a set function). (*)

(2) Lebesgue's singular function ψ is not abs. cont. We can enclose Cantor's middle third set in a union of pairwise disjoint open intervals (a_k,b_k) such that $\sum_{k=1}^{\infty}|b_k-a_k|$ is arbitrarily small. Extend ψ by setting $\psi(x)=0,\ x<0,\ \psi(x)=1,\ x>1$. Then it is easy to see that $\sum_{k=1}^{\infty}(\psi(b_k)-\psi(a_k))=1$ and hence $\sum_{k=1}^{n}(\psi(b_k)-\psi(a_k))\geq 1/2$ for suff. large n whereas $\sum(b_k-a_k)$ is arbitrarily small.

4.8. Thm. Let $f:[a,b]\to\mathbb{R}$ be abs. cont. and $f'(x)=0$ a.e. Then f is a const.

4.9. Thm. (Fundamentat thm of integral calculus for Lebesgue integr.) Let $f:[a,b]\to\mathbb{R}$ be abs. cont. Then $f'\in L^1([a,b])$ and

$$f(x)=f(a)+\int_a^x f'(t)\,dt \qquad \forall x\in[a,b].$$

4.10 Thm A function $f:[a,b]\to\mathbb{R}$ is of the form

$$f(x)=f(a)+\int_a^x \varphi(t)\,dt \quad \text{for some } \varphi\in L^1([a,b])$$

iff f is abs. cont. In this cese $\varphi(x)=f'(x)$ a.e. on $[a,b]$.

(*) If $f:[0,1]\to\mathbb{R}$ satisf. $\int|f|\,dm<\infty$ then $\forall\varepsilon>0\ \exists\delta>0$: $E\subset[0,1]$ $m(E)<\delta \implies \int_E|f|\,dm<\varepsilon$.

Let X be a metric space and $B \subset X$.

A family $\mathcal{A}$ of sets in X is a <u>cover</u> of B if $B \subset \cup\{A : A \in \mathcal{A}\}$. The family $\mathcal{A}$ is an <u>open</u> cover if every element in $\mathcal{A}$ is an open set. A sub-cover of $\mathcal{A}$ is a subfamily which also is a cover.

4.11. <u>Rmk.</u> As a rule, the set $\mathcal{A}$ need not be countable. Lindelöf has proved that there is a countable subcover of each open cover of a space whose topology has a countable subcover. In particular, this holds for all of our spaces $(\bar{\mathbb{R}}^n, q)$, $(\mathbb{R}^n, |\cdot|)$, (H^n, ρ) etc.

4.12. <u>Def.</u> A set $F \subset X$ is compact if every open cover of F has a finite subcover.

Next, pages 36–37 are discussed (they were previously omitted)

5. Capacity

Here, the capacity of a condenser is defined. A condenser is a generalization of a ring domain and its capacity is a generalization of the modulus of the family joining the bry components of a ring. One of the most important properties of the capacity its decrease under symmetrization of the condenser. It follows that two symmetric condensers — so called Grötzsch and Teichmüller condenser are extremal in character: their capacities yield lower bounds for a wide class of condensers.

One of the main themes of this section is to explore the connection between a real number — the capacity of a condenser — and the geometric structure of a condenser. We will further clarify the connection that we have proved to exist between $M(\Delta_{EF})$ and $\min\{d(E), d(F)\}/d(E,F)$ when E, F are connected.

5.1. Def. For $j = 1, \dots, n$ let $\mathbb{R}^n_j = \{x \in \mathbb{R}^n : x_j = 0\}$ and let $T_j : \mathbb{R}^n \to \mathbb{R}^n_j$ be the orthog. projection $T_j x = x - x_j e_j$. Let $D \subset \overline{\mathbb{R}}^n$ be an open set and $u : D \to \mathbb{R}$ cont. We say that $u : D \to \mathbb{R}$ is absol. contin on lines, abbr. ACL, if for $\forall$ cube Q, $\overline{Q} \subset D$, the set

$$A_j(Q) = \{z \in T_j Q : t \mapsto u(z + te_j),\ z + te_j \in Q, \text{ is not absolutely continuous as a function of one variable}\}$$

is of zero measure, $m_{n-1}(A_j(Q)) = 0$, $j = 1, \dots, n$.

According to basic results from real analysis of one variable, the deriv. of an absol. cont. function exists a.e. and is Borel measurable [HS, p. 285]. This fact and

Fubini's theorem implies that for an ACL function $u: D \to \mathbb{R}$, $D \subset \mathbb{R}^n$, all partial derivatives $\partial u(x)/\partial x_j$ exist a.e. (w.r.t. Lebesgue n-dim. measure) in D. We say that an ACL function $u: D \to \mathbb{R}$ is in ACL^p if $\partial u(x)/\partial x_j \in L^p(K)$, $j = 1, \ldots, n$ $\forall K$, K compact, $K \subset D$. We say that $f: D \to \mathbb{R}^m$, $f = (f_1, \ldots, f_m)$ is ACL (ACL^p) if f_j is, $\forall j$.

5.2. Def. Let $A \subset \mathbb{R}^n$ be open and $C \subset A$ compact. The pair $E = (A, C)$ is termed a condenser. Its p-capacity is defined by

$$p\text{-cap}\, E = \inf_{u \in \mathcal{F}} \int_{\mathbb{R}^n} |\nabla u|^p \, dm \tag{5.3}$$

where $\mathcal{F} = \{u : A \to \mathbb{R} : u \text{ is } ACL^p,\ u \geq 0,\ u(x) \geq 1\ \forall x \in C$ and $\operatorname{spt} u$ is compact$\}$. Here $\operatorname{spt} u = \overline{\{x : u(x) \neq 0\}}$ and $\nabla u(x) = (\partial u(x)/\partial x_1, \ldots, \partial u(x)/\partial x_n)$.

It is easy to see that p-cap E is invariant under translations and orthogonal maps. (In the case $p \neq n$, p-cap E is not invariant under stretchings!)

Without changing the real number p-cap E, one may replace the family $\mathcal{F}$ in (5.3) by several other families of functions. For instance we know that $\mathcal{F}$ can be replaced by $\mathcal{F}_1 = \{u : D \to \mathbb{R} : u \in C_0^\infty(A),\ u(x) \geq 1\ \forall x \in C$, $\operatorname{spt} u$ is compact, $\operatorname{spt} u \subset D\}$. (These results are in MRV1, GOR, MAZ2. The following monotonicity property of the capacity, which follows directly from the defin. will be used without special remark in what follows. If (A, C) and (A', C') are condensers