curve γ, denoted by $\ell_q(\gamma)$ [or $\ell_k(\gamma)$]. The quasih. distance between two points $x, y \in D$ is defined as

(2.27) $$k_D(x,y) = \inf_{\alpha \in \Gamma_{xy}} \ell_q(\alpha) = \inf_{\alpha \in \Gamma_{xy}} \int_\alpha w(x)\,|dx|$$

where $\Gamma_{xy} = \{\beta : \beta \text{ a rectifiable curve, } x, y \in \beta \subset \mathbb{R}^n\}$. It is easy to show that k_D is a metric. The defin. (2.26) gives immediately the invariance of k_D under translations, stretchings and orthogonal maps. The proof of the next theorem is omitted

2.28. Thm (Gehring–Osgood [GOS]) Given $x, y \in D$, there exists a geodesic segment $J_D[x,y]$ of k_D with

(1) $$\int_{J_D[x,y]} \frac{|dx|}{d(x,\partial D)} = k_D(x,y),$$

(2) $$\forall c \in J_D[x,y],\ k_D(x,c) + k_D(c,y) = k_D(x,y).$$

2.29. Rmk. Clearly $k_{H^n} = \rho_{H^n}$ and it is not difficult to show, that $\rho_{B^n} \le 2k_{B^n} \le 2\rho_{B^n}$. If D is given, it is usually impossible to find the geodesic segments. For domains $D_1 \subsetneq D_2 \subsetneq \mathbb{R}^2$ we have $k_{D_1}(x,y) \ge k_{D_2}(x,y)$ for all $x, y \in D_1$. A similar inequality also holds for j_D.

2.30. Lemma (Gehring–Palka [GP]) For $x, y \in D$
$$k_D(x,y) \ge j_D(x,y) \ge \left|\log \frac{d(x)}{d(y)}\right|.$$ [Proof omitted.]

For convenient use we record Bernoulli's inequality

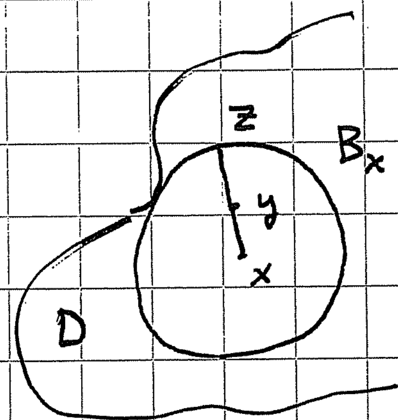
(2.31) $$\log(1+as) \le a\log(1+s);\quad a \ge 1,\ s > 0.$$

2.32. <u>Lemma</u>. (1) For $x \in D$ and $y \in B_x = B^n(x, d(x))$,

$$k_D(x,y) = \log\left(1 + \frac{|x-y|}{d(x) - |x-y|}\right).$$

(2) For $s \in (0,1)$, $|x-y| \le s\, d(x)$

$$k_D(x,y) \le \frac{1}{1-s}\, j_D(x,y)$$



<u>Proof</u>. (1) Fix $z \in \partial B_x$ s.t. $y \in [x,z]$.
Because $[x,y] \in \Gamma_{xy}$ we get by 2.28

$$k_D(x,y) \le k_{B_x}(x,y) \le \int_{[x,y]} \frac{|dw|}{d(w)} \le$$

$$\int_{[x,y]} \frac{|dw|}{|w-z|} = \int_{d(x)-|x-y|}^{d(x)} \frac{dt}{t} = \log \frac{d(x)}{d(x)-|x-y|} = \log\left(1 + \frac{|x-y|}{d(x)-|x-y|}\right).$$

(2) Apply (1) and Bernoulli ineq. (2.31) $\Longrightarrow$

$$k_D(x,y) \le \log\left(1 + \frac{|x-y|}{(1-s)d(x)}\right) \le \frac{1}{1-s} \log\left(1 + \frac{|x-y|}{d(x)}\right) \le \frac{1}{1-s} j_D(x,y).$$

Let $G \subsetneq \mathbb{R}^n$, $x \in G$, $M > 0$. The <u>quasih. ball</u> is

$$B_k = D_G(x,M) = \{z \in G : k_G(x,z) < M\}.$$

2.30 $\Longrightarrow$ $\forall z \in \overline{D}_G(x,M)$: $e^{-M} d(x) \le d(z) \le e^{M} d(x)$.

$$\left.\begin{array}{l} z \in B^n(x, (1-e^{-M})d(x)) \xrightarrow{2.32(1)} k_G(x,z) < M \\ z \in \mathbb{R}^n \setminus B^n(x, (e^M-1)d(x)) \xrightarrow{2.30} k_G(x,z) > M \end{array}\right\} \Longrightarrow$$

$$(2.33) \begin{cases} B^n(x, r\,d(x)) \subset D_G(x,M) \subset B^n(x, R\,d(x)) \\ \overline{D}_G(x,M) \subset \{z \in G : e^{-M} d(x) \le d(z) \le e^{M} d(x)\} \\ r = 1 - e^{-M}, \quad R = e^M - 1 \end{cases}$$

The numbers r and R are best possible as we see by considering the case $G = H^n$ (cf. (2.9))

According to Thm 2.20 Möbius transformations are hyperbolic isometries. The quasihyp. metric is not invariant in this sense. The following thm holds (proof omitted). 34

2.34. Thm If $G, G' \subsetneq \mathbb{R}^n$ and if $f: G \to fG = G'$ is a Möbius transformation, then for all $x, y \in G$

$$k_G(x,y)/2 \le k_{G'}(f(x), f(y)) \le 2\, k_G(x,y) .$$

2.35. Rmk. (1) The ineq. in 2.32(2) does not hold for all $x, y \in D$. To see this consider the domain $D = B^2 \setminus [0, e_1)$ and the points $x_t = (1/2, t)$, $y_t = (1/2, -t)$ in it. If there exists $C > 0$ such that for all $x, y \in D$ $k_D(x,y) \le C\, j_D(x,y)$, then the domain D is called uniform.

(2) J. Ferrand has given a GM-invariant version of the quasihyp. metric.

The hyperbolic volume of a Lebesgue measurable set $E \subset B^n$ is defined

$$m_h(E) = \int_E \frac{2^n\, dm(x)}{(1-|x|^2)^n} . \tag{2.36}$$

Integration in polar coordinates gives (ω_{n-1} is the $(n-1)$-dim. surface area of S^{n-1})

$$m_h(B^n(r)) = 2^n \omega_{n-1} \int_0^r \frac{t^{n-1}}{(1-t^2)^n}\, dt < \frac{2^n \omega_{n-1}}{n-1} \left(\frac{r}{1-r}\right)^{n-1} . \tag{2.37}$$

The last ineq. holds because

$$\int_0^r \frac{t^{n-1}\, dt}{(1-t^2)^n} \le r^{n-1} \int_0^r \frac{dt}{(1+t)^n (1-t)^n} < r^{n-1} \int_0^r \frac{dt}{(1-t)^n} = r^{n-1} \Big/_0^r - \frac{(1-t)^{-n+1}}{-n+1} .$$

For $s \in (1/2, 1)$ we have $t^{n-1} \geq 2^{2-n} t$, hence (2.37) $\Longrightarrow$

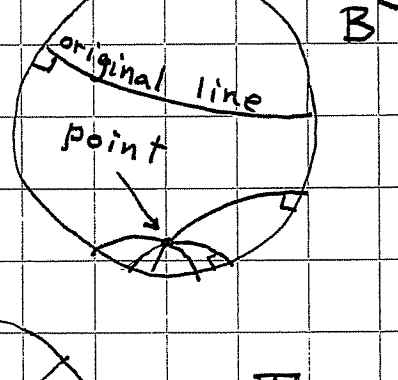
$$(2.38) \quad m_h(B^n(s)) > 2^n \omega_{n-1} 2^{2-n} \int_{1/2}^{s} \frac{t\,dt}{(1-t^2)^n} > c \Big/_{1/2}^{s} -\frac{1}{2}\frac{(1-t^2)^{1-n}}{1-n}$$

with $c = 2^{2(1-n)}\omega_{n-1}$. $GM(B^n)$-invariance ((2.17), (2.37)) $\Longrightarrow$

$$(2.39)\begin{cases} m_h(D(x,M)) = m_h(D(0,M)) = m_h\left(B^n\left(\operatorname{th}\frac{M}{2}\right)\right) \\ \leq \dfrac{2^n\omega_{n-1}}{n-1} \operatorname{th}^{n-1}(M/2)\left(1-\operatorname{th}\frac{M}{2}\right)^{1-n} \\ < \dfrac{2^n\omega_{n-1}}{n-1} e^{M(n-1)} \operatorname{th}^{n-1}(M/2). \end{cases}$$

$1 - \operatorname{th}\frac{M}{2} > e^{-M}$ is used here

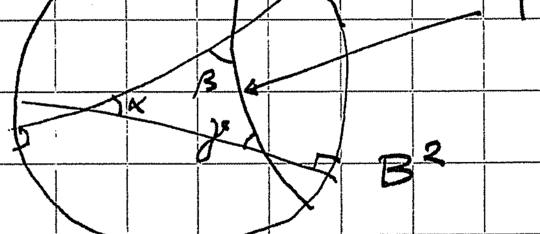
In the hyperb. geometry of B^2, the circular arcs $\perp \partial B^2$ are lines. The Parallel Postulate of the Euclidean geometry fails: given a line and a point not on it, there are ∞ many lines going through the point, not intersecting the orig. line.



The area of a hyperb. triangle T with angles α, β, γ is given by

$$m_2(T) = \pi - (\alpha+\beta+\gamma).$$

In particular, the sum of the angles of T is $< \pi$.



In math. analysis we sometimes need coverings by families of balls. The next theorem is an example.

2.40. Lemma. Let $(\mathbb{X}, d)$ be one of the metric spaces $(\mathbb{R}^n, |\,|)$, (B^n, ρ), (H^n, ρ) and $B_{\mathbb{X}}(z,r) = \{y \in \mathbb{X} : d(y,z) < r\}$. Let $A \neq \emptyset$, $A \subset \mathbb{X}$, $\mathcal{F} = \{B_{\mathbb{X}}(z, r(z)) : z \in A\}$ and let $\sup\{r(x) : x \in A\} < \infty$. Then there exists a number $c(n)$ (depending only on n) and a countable subfamily $\mathcal{F}_1 \subset \mathcal{F}$ s.t. (1) $A \subset \bigcup B_{\mathbb{X}}(z, r(z))$, $B_{\mathbb{X}}(z, r(z)) \in \mathcal{F}_1$ (2) each point $x \in A$ belongs to at most $c(n)$ elements of $\mathcal{F}_1$. [N.B. in part (1) the union is taken over all balls $B_{\mathbb{X}}(z, r(z))$ in $\mathcal{F}_1$. Because $\mathcal{F}_1$ is countable there are $x_k \in A$ such that $\mathcal{F}_1 = \{B_{\mathbb{X}}(x_k, r(x_k)) : k = 1, 2, \ldots\}$.]

Let $(\mathbb{X}, d)$ be a metric space $\emptyset \neq A \subset \mathbb{X}$ and set

$$(2.41) \quad p_{\mathbb{X}}(A, t) = \inf\{k : A \subset \bigcup_{j=1}^{k} B_{\mathbb{X}}(x_j, t),\ x_j \in A\}.$$

For a compact set A, $p_{\mathbb{X}}(A, t) < \infty$ for all $t > 0$. Because $\mathbb{X}$ is usually clear from the context, we often write $p(A,t) = p_{\mathbb{X}}(A, t)$.

2.42. Lemma. Let $(\mathbb{X}, d)$ be one of the metric spaces in Lemma 2.40, $m_{\mathbb{X}} = m$ for $\mathbb{X} = \mathbb{R}^n$, $m_{\mathbb{X}} = m_h$ for $\mathbb{X} = B^n$ or $\mathbb{X} = H^n$, and let $A \neq \emptyset$ be a compact subset of $\mathbb{X}$. Then

$$d_1\, m_X(A) \le p(A,t) \le c(n)\, d_1\, m_X\Big(\bigcup_{z\in A} B_X(z,t)\Big)$$

where $d_1 = 1/m_X(B_X(y,t))$ and $c(n)$ is as in 2.40.

Proof. Exercise

2.43. Exercise Show that for $X = B^n$

$$m_h\Big(\bigcup_{|x|\le r} D(x,M)\Big) \le d_2(n,M)(1-r)^{1-n}$$

$$p(B^n(r), M) \le d_1\, d_2\, (1-r)^{1-n}$$

II MODULUS AND CAPACITY

For disjoint sets $E, F \subset \overline{\mathbb{R}}^n$ let Δ_{EF} be the collection of all those curves that join E and F (i.e. if $\gamma \in \Delta_{EF}$ and $\gamma : [a,b] \to \overline{\mathbb{R}}^n$ then $\gamma(a) \in E$, $\gamma(b) \in F$ or $\gamma(a) \in F$, $\gamma(b) \in E$.) For a fixed set F the modulus of the family Δ_{EF}, $M(\Delta_{EF})$ defines an outer measure $m_F(E)$ in the set of all compact sets $\overline{\mathbb{R}}^n \setminus F$. This means that

(1) $m_F(\emptyset) = 0$

(2) $m_F(\bigcup_k E_k) \le \sum_{k=1}^{\infty} m(E_k)$ (subadditivity)

The real number $M(\Delta_{EF})$ gives quantitative information about the structure of the sets E and F and their position w.r.t. each other. Roughly speaking: $M(\Delta_{EF})$ is small if E and F are far from each other or if one of the sets E, F is "small" and large otherwise. In the important case when E and F are continua, one can can specify this statement:

$$\min\{d(E), d(F)\}/d(E,F) \quad \text{and} \quad M(\Delta_{EF})$$

are, at the same time, small/large. Many applications of the modulus rely on this fact and above all, on the conformal invariance of the modulus:

$$M(\Delta_{EF}) = M(\Delta_{h(E)\,h(F)}) \quad \forall\, h \in GM(\overline{\mathbb{R}}^n).$$

This invariance is crucial for what follows. We define e.g. two conformal invariants $\lambda_G(x,y)$ and $\mu_G(x,y)$, $x, y \in G$. Corresponding to the case of the modulus the numbers $\lambda_G(x,y)$ and $\mu_G(x,y)$ give information about the position of the points x, y w.r.t. each other and the boundary of G. One may think of $\mu_G(x,y)$ as a Möbius invar. metric and $\lambda_G(x,y)$ as its "dual" counterpart. Because μ_{B^n} can be expressed in terms of ρ_{B^n}, one may consider μ_{B^n} as a generalization of the hyperbolic metric ϱ_{B^n}.

<u>History of modulus:</u> Ahlfors - Beurling [AB] 1950, $n=2$ Fuglede [F] 1957, $n\geq 2$, Väisälä

<u>Books of qc/qr maps:</u> Lehto-Virtanen [LV2], Ahlfors [A2] ($n=2$) Väisälä [V7], Caraman [C1], Reshetnyak [R12], Vuorinen CGQM 1988, Rickman 1992, Heinonen-Kilpeläinen-Martio 1992 ($n\geq 2$).

3. The modulus of a curve family

A curve γ is a continuous map of an interval $\Delta\subset\mathbb{R}$, $\gamma:\Delta\to\overline{\mathbb{R}}^n$. Often we identify γ and $\gamma\Delta$. Sometimes we set $|\gamma|=\gamma\Delta$ (the locus or trace of γ). The length of γ is denoted $\ell(\gamma)$ and defined in the usual way, using a polygonal approximation and passage to limit. If $\ell(\gamma)<\infty$ then γ is called rectifiable and γ has a parametrization in terms of arc length which is Lip-continuous. If γ is rectifiable and γ^0 is its parametrization in terms of the arc length, we may define the <u>line integral</u>

$$\int_\gamma f\,ds=\int_0^{\ell(\gamma)} f(\gamma^0(t))\,|\gamma^{0\prime}(t)|\,dt$$

whenever f is Lebesgue measurable on $[0,\ell(\gamma)]$. If γ is only locally rectifiable, then we set

$$\int_\gamma f\,ds = \sup\{\int_\alpha f\,ds : \alpha\subset\gamma,\ \alpha \text{ rectifiable}\}$$

For a curve family Γ in $\mathbb{R}^n$, we write $F(\Gamma)=\{\varrho:\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}:$ ϱ is Borel measurable, $\varrho\geq 0$ and $\int_\gamma \varrho\,ds\geq 1$ for all locally rectifiable $\gamma\in\Gamma\}$. When $p\geq 1$ we define the <u>p-modulus</u> of Γ

$$(3.1)\qquad M_p(\Gamma)=\inf_{\varrho\in F(\Gamma)}\int \varrho^p\,dm$$

Here m is the Lebesgue measure in $\mathbb{R}^n$ and $\int\varrho^p\,dm=\int_{\mathbb{R}^n}\varrho^p\,dm$. If $F(\Gamma)=\emptyset$ we set $M_p(\Gamma)=\infty$. Usually $p=n$ and we write

$M(\Gamma)$ instead of $M_n(\Gamma)$. If $M_p(\Gamma)>0$, the number $M_p(\Gamma)^{1/(1-p)}$ is called the extremal length of Γ. The set $F(\Gamma)$ is the set of all admissible functions for Γ.

3.2. Lemma M_p is an outer measure:

(1) $M_p(\emptyset)=0$,

(2) $\Gamma_1 \subset \Gamma_2 \Rightarrow M_p(\Gamma_1) \le M_p(\Gamma_2)$

(3) $M_p(\bigcup_{k=1}^{\infty} \Gamma_k) \le \sum_{k=1}^{\infty} M_p(\Gamma_k)$

Proof. (1) $f \in L^p(\mathbb{R}^n)$, f Borel $\Rightarrow |f| \in F(\emptyset) \Rightarrow M_p(\emptyset) \le \|f\|_p^p$ $\forall f \Rightarrow M_p(\emptyset)=0$.

(2) $F(\Gamma_2) \subset F(\Gamma_1)$

(3) Let $\varepsilon>0$. Choose $\varrho_k \in F(\Gamma_k)$: $\int \varrho_k^p \, dm \le M_p(\Gamma_k) + \frac{\varepsilon}{2^k}$ $\Rightarrow \varrho = (\sum \varrho_k^p)^{1/p} \in F(\bigcup_{k=1}^{\infty} \Gamma_k)$.

Let Γ_1 and Γ_2 be curve families in $\mathbb{R}^n$. We say that Γ_1 minorizes Γ_2, denoted $\Gamma_1 < \Gamma_2$ if every $\gamma \in \Gamma_2$ has a subcurve $\gamma_1 \in \Gamma_1$. The families $\Gamma_1, \Gamma_2, \ldots$ are called separate if there are disjoint Borel sets $E_k \subset \mathbb{R}^n$ such that if $\gamma \in \Gamma_k$ is loc. rectifiable, then $\int_\gamma g_k \, ds = 0$ where $g_i = \chi_{\mathbb{R}^n \setminus E_k}$.

3.3. Lemma. $\Gamma_1 < \Gamma_2 \Rightarrow M_p(\Gamma_1) \ge M_p(\Gamma_2)$.

Proof. $F(\Gamma_1) \subset F(\Gamma_2)$.

3.4. Lemma. If Γ_k are separate and $\Gamma < \Gamma_k$ $\forall k$, then $M_p(\Gamma) \ge \sum M_p(\Gamma_k)$.

Proof. For $\varrho \in F(\Gamma)$ set $\varrho_k(x) = (1-g_k(x))\varrho(x)$. Then $\varrho_k \in F(\Gamma_k)$ and

$$\sum M_p(\Gamma_k) \le \sum \int \varrho_k^p \, dm = \sum \int_{E_k} \varrho^p \, dm \le \int \varrho^p \, dm$$

Thus $\sum M_p(\Gamma_k) \le M_p(\Gamma)$.

3.5. Lemma Let Γ be a family of rectifiable curves, not containing constant curves, such that for a Borel set G with $m(G)=0$ all curves γ are subsets of G. Then $M_p(\Gamma)=0$.

Proof. Let $\Gamma_k = \{\gamma\in\Gamma : \ell(\gamma)\geq 1/k\}$, $k=1,2,\ldots$ Then $\Gamma=\cup\Gamma_k$ and $k\chi_G \in F(\Gamma_k)$. Now for all k

$$M_p(\Gamma_k) \leq \int k^p \chi_G = 0$$

implying that $M(\Gamma)=0$ by Lemma 3.2(3).

3.6. Theorem. If Γ is a family of curves in $\mathbb{R}^n$ and $\Gamma_r = \{\gamma\in\Gamma : \ell(\gamma)<\infty\}$, then $M(\Gamma)=M(\Gamma_r)$.

Proof. See Väisälä [V7, 6.9]

For $E,F,G\subset\overline{\mathbb{R}}^n$ let $\Delta(E,F;G)=\{\gamma : [a_\gamma, b_\gamma]\to\overline{\mathbb{R}}^n :$ $\gamma(a_\gamma)\in E,\ \gamma(b_\gamma)\in F,\ \gamma(t)\in G\ \forall t\in(a_\gamma,b_\gamma),\ \gamma$ non-constant$\}$.
Abbreviation: $\Delta(E,F)=\Delta(E,F;\overline{\mathbb{R}}^n)$.

3.7. Rmk. (1) If $E=\bigcup_{k=1}^{\infty} E_k \subset G$, $F\subset G$, then

$$M(\Delta(E,F;G)) \leq \sum_{k=1}^{\infty} M(\Delta(E_k,F;G))$$

(2) Let $D\subset\overline{\mathbb{R}}^n$ be open and $F\subset D$. By 3.2(2)

$$M_p(\Delta(F,\partial D; D\setminus F)) \leq M_p(\Delta(F,\partial D; D)) \leq M_p(\Delta(F,\partial D;\overline{\mathbb{R}}^n)) = M_p(\Delta(F,\partial D))$$

On the other hand $\Delta(F,\partial D;D) < \Delta(F,\partial D)$ and $\Delta(F,\partial D; D\setminus F) < \Delta(F,\partial D; D)$ and thus by 3.3

$$M_p(\Delta(F,\partial D)) = M_p(\Delta(F,\partial D; D)) = M_p(\Delta(F,\partial D; D\setminus F)) \tag{3.8}$$

(3) If $G\subset\mathbb{R}^n$ is Borel, $r>0$, $\Gamma=\{\gamma : \gamma\subset G,\ \ell(\gamma)\geq r\}$ then

$$M_p(\Gamma) \leq m(G)\, r^{-p}.$$

(4) The best source for modulus is Väisälä's excellent book [V7]. We give here, without proof, some results from [V7].

As the definition (3.1) suggests, for a given Γ it is usually difficult/impossible to find $M_p(\Gamma)$. In fact, $M_p(\Gamma)$ is known only for a few curve families Γ. In some applications it is enough to know some estimates of $M(\Gamma)$ instead of the exact value.

3.9. The cylinder. Let E be a Borel set in R^{n-1} and let $h > 0$. Set

$$G = \{x \in R^n \mid (x_1, \ldots, x_{n-1}) \in E \text{ and } 0 < x_n < h\}.$$

Then G is a cylinder with bases E and $F = E + he_n$ and with height h. Set $\Gamma = \Delta(E, F, G)$. We show that

$$M_p(\Gamma) = \frac{m_{n-1}(E)}{h^{p-1}} = \frac{m(G)}{h^p}.$$

Since $\ell(\gamma) \geq h$ for every $\gamma \in \Gamma$, 3.7(3) implies $M_p(\Gamma) \leq m(G)/h^p$. Let ϱ be an arbitrary function in $F(\Gamma)$. For each $y \in E$ let $\gamma_y : [0, h] \to R^n$ be the vertical segment $\gamma_y(t) = y + te_n$. Then $\gamma_y \in \Gamma$. Assuming that $p > 1$ we obtain by Hölder's inequality

$$1 \leq \left(\int_{\gamma_y} \varrho\, ds\right)^p \leq h^{p-1} \int_0^h \varrho(y+te_n)^p\, dt .$$

Integration over $y \in E$ yields by Fubini's theorem

$$m_{n-1}(E) \leq h^{p-1} \int_E dm_{n-1} \int_0^h \varrho(y+te_n)^p\, dt = h^{p-1} \int_G \varrho^p\, dm \leq h^{p-1} \int \varrho^p\, dm .$$

Since this holds for every $\varrho \in F(\Gamma)$, we obtain $M_p(\Gamma) \geq$ $m_{n-1}(E)/h^{p-1}$.

The proof for $p = 1$ is somewhat simpler. Δ

3.10. Remark. The above proof also shows that $M_p(\Gamma) = M_p(\Gamma_0)$ where Γ_0 is the subfamily of Γ consisting of the vertical segments γ_y.

3.11. Remark. In Example 3.9, $M_p(\Gamma)$ is invariant under similarity mappings iff $p = n$. This is the reason why the case $p = n$ is so important in the theory of qc mappings. Indeed, we shall show in the next section that $M(\Gamma)$ is a conformal invariant.

3.12. The spherical ring. If $0 < a < b < \infty$, the domain $A = B^n(b) \setminus \bar{B}^n(a)$ is called a spherical ring. Let $E = S(a)$, $F = S(b)$ and $\Gamma_A = \Delta(E, F, A)$. We shall prove that

$$M(\Gamma_A) = \omega_{n-1} (\log \frac{b}{a})^{1-n} .$$

Let $\varrho \in F(\Gamma_A)$. For each unit vector $y \in S^{n-1}$ we let $\gamma_y : [a, b] \to R^n$ be the radial segment, defined by $\gamma_y(t) = ty$. By Hölder's inequality we obtain

$$1 \le (\int_{\gamma_y} \varrho \, ds)^n \le \int_a^b \varrho(ty)^n t^{n-1} \, dt \, (\int_a^b t^{-1} \, dt)^{n-1}$$

$$= (\log \frac{b}{a})^{n-1} \int_a^b \varrho(ty)^n t^{n-1} \, dt .$$

Integrating over $y \in S^{n-1}$ yields

(3.13) $$\omega_{n-1} \le (\log \frac{b}{a})^{n-1} \int \varrho^n \, dm .$$

Taking the infimum over all $\varrho \in F(\Gamma)$ we obtain

$$\omega_{n-1} \le (\log \frac{b}{a})^{n-1} M(\Gamma_A) .$$

On the other hand, we have equality in (3.13) if we define $\varrho(x) = 1/|x| \log(b/a)$ for $x \in A$ and $\varrho(x) = 0$ otherwise. It is easy to see that this ϱ belongs to $F(\Gamma)$.

3.14. Remark. Let Y be a Borel set in S^{n-1} and let C be the cone $\{x \in R^n \mid x/|x| \in Y\}$. Set $\Gamma = \{\gamma \in \Gamma_A \mid |\gamma| \subset C\}$ where A is as above. Then the method of 3.12 yields

$$M(\Gamma) = m_{n-1}(Y)(\log \frac{b}{a})^{1-n}.$$

In fact, $M(\Gamma) = M(\Gamma_o)$ where Γ_o is the family of all radial segments γ_y, $y \in Y$.

3.15 The degenerate ring. Let $\Gamma = \Delta(E, F, G)$ where $E = \{0\}$, $F = S^{n-1}(b)$ and $G = B^n(b) \setminus \{0\}$. Since $\Gamma > \Gamma_A$ for every spherical ring $A = B^n(b) \setminus \bar{B}^n(a)$, we obtain from 3.3 and 3.12

$$M(\Gamma) \leq M(\Gamma_A) \leq \omega_{n-1}(\log \frac{b}{a})^{1-n}.$$

Since this holds for every $a > 0$, $M(\Gamma) = 0$.

3.16 Paths through a point. Let $x_o \in \bar{R}^n$ and let Γ be the family of all non-constant paths γ such that $x_o \in |\gamma|$. We show that $M(\Gamma) = 0$. If $x_o = \infty$, this is trivial. If $x_o \neq \infty$, we let Γ_k be the family of all $\gamma \in \Gamma$ such that $|\gamma|$ meets $S(x_o, 1/k)$. Then $M(\Gamma_k) = 0$ by 3.15 and 3.3. On the other hand, $\Gamma = \cup \Gamma_k$, whence by 3.2, $M(\Gamma) \leq \sum M(\Gamma_k) = 0$. Hence, given $x_o \in \bar{R}^n$, almost every non-constant path omits x_o.

3.17. Conformal mappings. Let D, D' be domains in $\mathbb{R}^n$ and $f: D \to D'$ a C^1-homeo. We say that f is conformal, if $|f'(x)h| = |f'(x)||h|$ $\forall x \in D$, $h \in \mathbb{R}^n$ and $J_f(x) \neq 0$ $\forall x \in D$. If $D, D' \subset \overline{\mathbb{R}}^n$ and $f: D \to D'$ is a homeo, then we say that f is conformal if $f|D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal in the above sense.

3.18. Rmk. It is not difficult to show that if $f: D \to D'$ and $g: D' \to D''$ are conf. then so are $g \circ f$ and f^{-1}. GM $\subset$ conformal.

3.19. <u>Lemma</u> Let $f: D \to D'$ be conformal. Then $M(f\Gamma) = M(\Gamma)$ for all Γ in D.

<u>Proof</u>. Fix $\varrho' \in F(\Gamma')$ and write $\varrho = \varrho'(f(x))|f'(x)|\chi_D$. Then

$$\int_\gamma \varrho \, ds = \int_{f\circ\gamma} \varrho' \, ds \geq 1 \quad \forall \text{ loc. rectif. } \gamma \in \Gamma$$

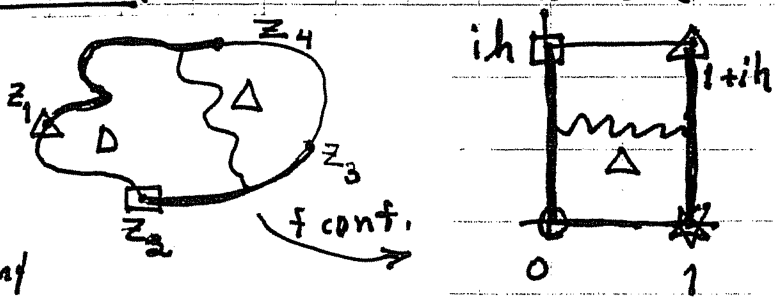
implying $\varrho \in F(\Gamma)$ and therefore

$$M(\Gamma) \leq \int \varrho^n \, dm = \int_D \varrho'(f(x))^n \underbrace{|f'(x)|^n}_{J_f(x)} dm = \int_{D'} (\varrho')^n \, dm$$

Because this holds for all $\varrho' \in F(\Gamma')$ we see $M(\Gamma) \leq M(\Gamma')$. The same argument also applies to f^{-1}, yielding $M(\Gamma') \leq M(\Gamma)$.

3.20. <u>Rmk.</u> According to Riemann's mapping theorem there are plenty of conformal maps in the plane: given two simply-conn. domains $D, D' \subset \mathbb{R}^2$ (with boundary $\neq$ point) there exists a conformal map $f: D \to D' = fD$. According to Liouville's theorem the situation in the space is different: a conformal mapping $f: D \to D'$, $D, D' \subset \mathbb{R}^n$, $n \geq 3$, $f \in C^3(D)$ is of the form $g|D$, $g \in GM(\overline{\mathbb{R}}^n)$.

3.21. <u>Modulus of a quadrilateral</u>. Recall the modulus of a quadrilateral from p.4



- D Jordan domain
- z_1, z_2, z_3, z_4 in posit. order on ∂D; $f: D \to [0,1] \times [0,h]$ conf

$M(\Delta, z_1, z_2, z_3, z_4) = h$

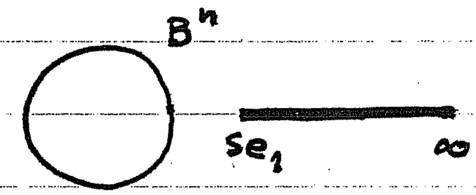
Let Δ be the family of curves in the rectangle $[0,1] \times [0,h]$

joining the vertical sides. Then by 3.9

$$M(\Delta) = \frac{1 \cdot h}{1^2} = h$$

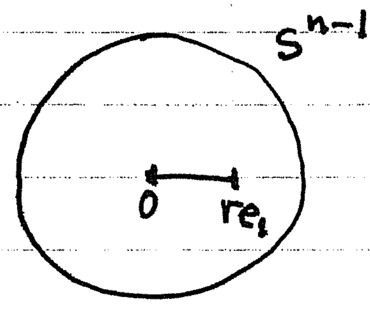
If $\Gamma = f^{-1}\Delta$ then $M(\Gamma) = M(\Delta) = h$ by 3.19. In conclusion, the modulus $M(D; z_1, z_2, z_3, z_4)$ is equal to the modulus of the family of curves joining in D the bry arcs between z_4, z_1 and z_2, z_3.

3.22. <u>Grötzsch ring</u>. A domain $D \subset \overline{\mathbb{R}}^n$ whose complement in $\overline{\mathbb{R}}^n$ has two components is called <u>a ring</u>. A ring with complementary components E and F is denoted $R = R(E, F)$. The <u>Grötzsch ring</u> is $R(\overline{B}^n, [se_1, \infty))$, $s > 1$. The <u>bounded G ring</u> is $R(\overline{\mathbb{R}}^n \setminus B^n, [0, re_1])$, $r \in (0,1)$.



These are examples of canonical ring domains, important for qc maps.

For $n = 2$ the bounded G ring $R_G(r)$ The Grötzsch ring can be mapped onto the annulus $B^2 \setminus \overline{B}^2(t)$ by a conf. mapping.



Conformal invariance $\Rightarrow M(\Gamma) = M(\Gamma')$

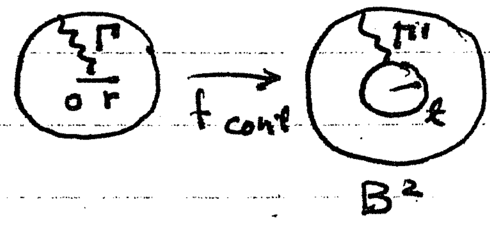
It can be shown that

(3.23) $M(\Gamma) = 2\pi / \mu(r) = 2\pi / \log \frac{1}{r}$

where

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}; \quad \mathcal{K}(r) = \int_0^1 [(1-x^2)(1-r^2x^2)]^{-1} dx$$

The bounded G ring



The function μ satisfies the following functional identities where $r' = \sqrt{1-r^2}$, $0 < r < 1$

$$(3.24) \begin{cases} \mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right) = \pi^2/(4\mu(r')) \\ \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(1/\sqrt{2}) = \pi/2 \end{cases}$$