(2) For distinct $a,b,c,d \in \overline{\mathbb{R}}^n$ and $t = |a,b,c,d|$ $\exists f \in GM$ such that $f(a)=0$, $f(b)=e_1$, $|f(c)| = t$, $f(d) = \infty$. Thus the quadruple (a,b,c,d) may be normalized such that $(a,b,c,d) \longmapsto (0, e_1, z, \infty)$ where $|z| = |a,b,c,d|$.

(3) The quadruple $(0, e_1, z, \infty)$ is Möbius equivalent to $(-e_1, y, -y, e_1)$ where $|y| \le 1$. For this see AVVB 7.24.

1.25. The group $M(B^n)$. A canonical representation for an element of $M(B^n)$ is given. We write

(1.26) $\quad a^* = a/|a|^2;\ a \in \mathbb{R}^n \setminus \{0\},\ 0^* = \infty,\ \infty^* = 0.$

Let $a \in B^n \setminus \{0\}$ be given. We wish to find $f \in M(B^n)$ with $f(a) = 0$. Let

(1.27) $\quad \sigma_a(x) = a^* + r^2 (x - a^*)^*;\ r^2 = |a|^{-2} - 1.$

Then σ_a is an inversion in the sphere $S^{n-1}(a^*, r)$ which is $\perp$ to S^{n-1}. Let p_a be the reflection in the $(n-1)$-dim. plane $P(a,0)$, which goes through the origin and is $\perp a$. Then $p_a \circ \sigma_a$ is op and thus $T_a \equiv p_a \circ \sigma_a \in M(B^n)$. Because $S^{n-1} \perp S^{n-1}(a^*, r)$ it follows that $T_a \in M(B^n)$. Furthermore

$T_a(a) = 0, \quad T_a(\mp a/|a|) = \mp a/|a|$

Thus $T_a[-a/|a|, a/|a|] = [-a/|a|, a/|a|]$. For $a = 0$ we set $T_a \equiv id$. This T_a is the desired map.

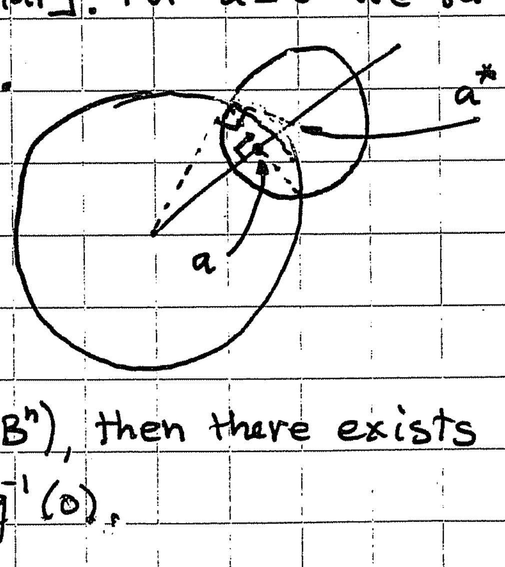


Figure The action of σ_a

We often write gx pro $g(x)$.

The canonical representation is:

1.28. Thm (Beardon p. 40) If $g \in GM(B^n)$, then there exists $k \in O(n)$ s.t. $g = k \circ T_a$ where $a = g^{-1}(0)$.

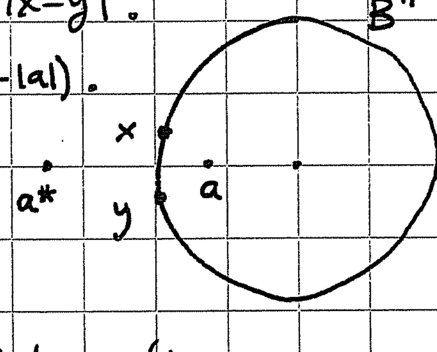
1.29. <u>The Lipschitz constant of $T_a|B^n$</u>. Let $T_a = p_a \circ \sigma_a$ be as above. Because p_a is a reflection and therefore euclidean isometry, we have by formula (1.4)

$|T_a x - T_a y| = |\sigma_a x - \sigma_a y| = r^2 |x-y| / (|x-a^*||y-a^*|)$.

By simple geometry $|z-a^*| \in [1/|a| - 1, 1/|a| + 1]$ for all $z \in B^n$. Therefore for all $x, y \in B^n$ ($r^2 = |a|^{-2} - 1$)

$$|T_a x - T_a y| \le \left(\frac{|a|}{1-|a|}\right)^2 (|a|^{-2} - 1)\, |x-y| \le \frac{1+|a|}{1-|a|} |x-y|.$$

In conclusion: $\mathrm{Lip}(T_a | B^n) \le (1+|a|)/(1-|a|)$.



On the other hand, utilizing (1.4) one can show that

$$(1.30) \qquad \mathrm{Lip}(T_a | B^n) = \frac{1+|a|}{1-|a|}.$$

To this end, fix $x, y \in S^{n-1}$ as in the picture (hence $[x,y] \perp [a, a^*]$) so that $|x-a^*| = |y-a^*|$, and let $|x-y| \to 0$. Because $T_a^{-1} = T_{-a}$ we see that $T_a | B^n$ is bilip with the constant (1.30).

1.31. <u>Exer.</u> Let $0 < s < 1$. Apply (1.4) to show that $\forall x, y, a \in B^n(s)$

$$\frac{1-s^2}{(1+s^2)^2} |x-y| \le |T_a x - T_a y| \le \frac{1}{1-s^2} |x-y|.$$

<u>$Q(z,r)$ as a set.</u> The ball $Q(z,r)$, $z \in \mathbb{R}^n$, $r \in (0,1)$ is set of the following kind a) Eucl. ball $B^n(a,s)$, b) $\overline{\mathbb{R}}^n \setminus \overline{B}^n(b,t)$ c) a halfspace of $\mathbb{R}^n$ whose boundary is an $(n-1)$-dim. hyperplane T. The case a) occurs iff $\infty \in \overline{\mathbb{R}}^n \setminus Q(z,r)$, b) iff $\infty \in Q(z,r)$ and c) iff $\infty \in \partial Q(z,r)$.

A sphere Σ in $\overline{\mathbb{R}}^n$ is a set of the form $\partial Q(z,r)$ (previously we have considered spheres in $\mathbb{R}^n$). A reflection in Σ is defined as follows. If $\Sigma = S^{n-1}(a,s)$ then a reflection in Σ is an Inversion in $S^{n-1}(a,s)$. If Σ is an $(n-1)$-dim. plane T, then reflection in Σ is the reflection in T.

1.32. Thm (Beardon p. 31) Let Σ be a sphere in $\overline{\mathbb{R}}^n$, σ a reflection in Σ and I the identity map. If $f \in GM(\overline{\mathbb{R}}^n)$ and $f(x) = x$ for all $x \in \Sigma$, then $f = I$ or $f = \sigma$.

1.33. The Ahlfors bracket. Basic algebra gives (recall $x^* = x/|x|^2$)

$$|x|^4 |y - x^*|^2 = (y|x|^2 - x)\cdot(y|x|^2 - x) = |x|^2 (1 - 2x\cdot y + |x|^2|y|^2),$$

and hence $|x|^2 |y - x^*|^2 = 1 - 2x\cdot y + |x|^2|y|^2$. The Ahlfors bracket $A[x,y]$ is defined by

$$(1.34)\quad A[x,y]^2 = 1 - 2x\cdot y + |x|^2|y|^2 = |x|^2|y - x^*|^2$$
$$= (1-|x|^2)(1-|y|^2) + |x-y|^2 .$$

Because p_a is a Euclidean isometry and $T_a = p_a \circ \sigma_a$ we have

$$(1.35)\quad |T_a x| = |T_a x - \underbrace{T_a a}_{=0}| = |\sigma_a x - \sigma_a a| = \frac{r^2 |x-a|}{|x-a^*||a-a^*|} = \frac{|x-a|}{|a||x-a^*|} = \frac{|x-a|}{A[x,a]}$$

where we used $r^2 = |a|^{-2}(1-|a|^2)$, $|a-a^*| = (1-|a|^2)/|a|$.

By (1.34) and (1.35) we obtain

$$(1.36)\quad 1 - |T_a x|^2 = 1 - \frac{|x-a|^2}{A[x,a]^2} = \frac{(1-|a|^2)(1-|x|^2)}{A[x,a]^2} .$$

Now we can also write

$$(1.37)\quad \sigma_a x = a^* + r^2 (x - a^*)^* = a^* + (1-|a|^2)\frac{x-a^*}{A[x,a]^2} .$$

1.38. Lemma For $a \in B^n$, $x \in \mathbb{R}^n$, $A = A[x,a]$ we have

$$T_a x = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{A^2}$$

Proof. Because $p_a z = z - 2(a\cdot z)$ and $T_a = p_a \circ \sigma_a$ we get by (1.37)

$$A^2 T_a x = A^2 p_a \sigma_a x = a^* A^2 + (1-|a|^2)(x - a^*) - 2(A^2 + (1-|a|^2)(a\cdot x - 1))a^*$$

where we used $a\cdot a^* = 1$. Further,

$$A^2 T_a x = a^*\left(-A^2 - 2(1-|a|^2)(a\cdot x-1)\right) + (1-|a|^2)(x-a^*)$$
$$= a^*\left(-|x|^2|a|^2 + 2a\cdot x - 1 - 2(1-|a|^2)(a\cdot x-1) - (1-|a|^2)\right) + (1-|a|^2)x$$
$$= a^*\left(-|x|^2|a|^2 + 2|a|^2 a\cdot x - |a|^2\right) + (1-|a|^2)x$$

Thus
$$T_a x = -\frac{a}{A^2}\left(|x|^2 - 2a\cdot x + 1\right) + \frac{(1-|a|^2)x}{A^2} = \frac{(1-|a|^2)(x-a) - |x-a|^2 a}{A^2}.$$

1.39. Rmk. For $x, y \in B^n$ let $|x| = r$, $|y| = s$, $|x-y| = d$ and $r' = \sqrt{1-|x|^2}$, $s' = \sqrt{1-s^2}$. Then by AVVB 7.57 (1)
$$d + (1-r)(1-s) \le A[x,y] \le d + r's'.$$

1.40. Spherical isometries. For a given $z \in \mathbb{R}^n \setminus \{0\}$ we wish to find a spherical isometry s_z such that $s_z(z) = 0$. Let f_z be the inversion in $S^{n-1}(-z/|z|^2, \sqrt{1+|z|^{-2}})$, i.e.
$$f(x) = -\frac{z}{|z|^2} + \frac{(1+|z|^{-2})(x + z/|z|^2)}{|x + z/|z|^2|^2}.$$
We calculate
$$f(z) = -\frac{z}{|z|^2} + \frac{(1+|z|^{-2})\, z\, (1 + 1/|z|^2)}{|z|^2 (1 + 1/|z|^2)^2} = 0.$$

Claim. f is a q-isometry

Proof. Write $\tilde z = -z/|z|^2$. By (1.4)
$$|f(x) - f(y)| = \frac{(1+|z|^{-2})|x-y|}{|x-\tilde z||y-\tilde z|} = \xi$$

Again by (1.4): $|f(x)| = |f(x) - \underbrace{f(z)}_{=0}| = \dfrac{(1+|z|^{-2})|x-z|}{|x-\tilde z||z+\tilde z|} = \dfrac{|x-z|}{|z||x-\tilde z|}$

Similarly: $|f(y)| = \dfrac{|y-z|}{|z||y-\tilde z|}$.

These formulas yield

(a) $$q(fx,fy)=\frac{|fx-fy|}{\sqrt{1+|fx|^2}\sqrt{1+|fy|^2}}=\frac{\xi}{\sqrt{1+\frac{|x-z|^2}{|z|^2|x-\tilde{z}|^2}}\sqrt{1+\frac{|y-z|^2}{|z|^2|y-\tilde{z}|^2}}}$$

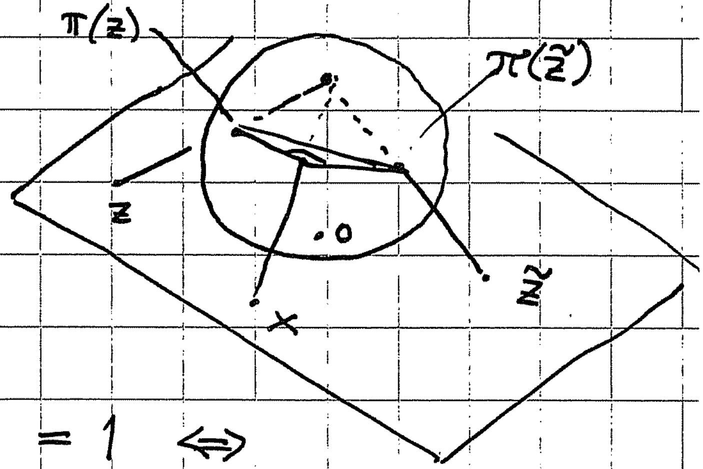
$$=\frac{(1+|z|^2)\,|x-y|}{\sqrt{|z|^2|x-\tilde{z}|^2+|x-z|^2}\sqrt{|z|^2|y-\tilde{z}|^2+|y-z|^2}}\,.$$

Recall (1.18) in the form

$q(x,z)^2+q(x,\tilde{z})^2=1.$

This is the same as the Pythagorean theorem

$|\pi x-\pi z|^2+|\pi x-\pi\tilde{z}|^2=1.$



$$\Longleftrightarrow \frac{|x-z|^2}{(1+|x|^2)(1+|z|^2)}+\frac{|x-\tilde{z}|^2}{(1+|x|^2)(1+|z|^{-2})}=1 \Longleftrightarrow$$

(b) $$\begin{cases} 1+|x|^2=\dfrac{|x-z|^2}{1+|z|^2}+\dfrac{|x-\tilde{z}|^2}{1+|z|^{-2}}=\dfrac{|x-z|^2+|z|^2|x-\tilde{z}|^2}{1+|z|^2} \\ 1+|y|^2=\dfrac{|y-z|^2+|z|^2|y-\tilde{z}|^2}{1+|z|^2} \end{cases}$$

Substitution of (b) in (a) yields

$$q(fx,fy)=\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}=q(x,y)\,,$$

i.e. f really is a q-isometry and we can choose $s_z=f$.

<u>The isometry t_z.</u> Let p_z be a reflection in the plane $P(z,0)\ni 0$. We set (here s_z is as above)

(1.41) $\qquad t_z=p_z\circ s_z\,.$

Now t_z is op and hence $t_z\in M(\overline{\mathbb{R}}^n)$ and $t_z(z)=0$.

If $z=0$ we set $t_z=id$, if $z=\infty$ we set $t_z=p_1\circ s$ where s in the inversion in S^{n-1} and p_1 reflection in the plane $\{x\in\mathbb{R}^n:\ x_1=0\}$. Because t_z is a q-isom.

$$(1.42)\quad \begin{cases} \mathfrak{t}_z Q(z,r) = Q(0,r) = B^n(r/\sqrt{1-r^2}), & x,z \in \overline{\mathbb{R}}^n, \\ |\mathfrak{t}_z(x)|^2 = q(x,z)^2/(1-q(x,z)^2), & r \in (0,1). \end{cases}$$

1.43. <u>Thm.</u> The formula (1.41) defines a q-isometry $\mathfrak{t}_z \in M(\overline{\mathbb{R}}^n)$ with $\mathfrak{t}_z(z)=0$. Furthermore $h \in GM(\overline{\mathbb{R}}^n)$ is a q-isometry iff $\mathfrak{t}_{h(0)} \circ h \in O(n)$.

<u>Proof.</u> It is enough to prove the statements about h. The map $f = \mathfrak{t}_{h(0)} \circ h$ satisfies $f(0)=0$ which yields (because f is a q isometry, too) that $fB^n = B^n$ and thus $f \in GM(B^n)$. Thm 1.28 shows that $f \in O(n)$. "The other direction" is clear.

1.44. <u>Thm.</u> Let $a \in \mathbb{R}^n$, $r>0$ and $b \in \mathbb{R}^n$, $u>0$ s.t. $B^n(a,r) = Q(b,u)$. If f is an inversion in $S^{n-1}(a,r)$, then

$$f = \mathfrak{t}_b^{-1} \circ f_1 \circ \mathfrak{t}_b$$

where $\mathfrak{t}_b$ is as in (1.41) and f_1 is the inv. in $S^{n-1}\left(\frac{u}{\sqrt{1-u^2}}\right)$.

<u>Proof.</u> Write $g = \mathfrak{t}_b^{-1} \circ f_1 \circ \mathfrak{t}_b$. Then $g(z)=z$ for $z \in S^{n-1}(a,r)$. On the other hand g is or, hence $g \neq id$. Thm 1.32 $\Rightarrow$ $g = f$.

1.45. <u>Distances under inversion.</u> We next give a counterpart of (1.4) for the q-metric.

1.46. <u>Thm.</u> Let $B^n(a,t) = Q(b,r)$, $a,b \in \mathbb{R}^n$, $t>0$, $r \in (0,1)$ and let g be an inversion in $S^{n-1}(a,t)$. Then

$$(1.47)\quad q(g(x),g(y)) = \frac{r^2 (r')^2 q(x,y)}{H(x)\,H(y)}$$

where $H(x)=\sqrt{(r')^4 q(x,b)^2 + r^4 q(x,\tilde{b})^2}$ and $\tilde{b} = -b/|b|^2$.

Proof. Because both sides of (1.47) are invariant under q-isometries we may assume, by making auxiliary maps of this type that $b=0$ and $g(b)=\infty$. By 1.19 $\partial Q(0,r) = S^{n-1}(r/\sqrt{1-r^2})$ and because $g(\infty)=b=0$, we obtain from (1.4) $|g(x)-g(\infty)| = |g(x)| = (r/r')^2 \sqrt{1-u^2}/u$, $u=q(x,0)$.
We see that

$$\frac{|x-y|}{|x||y|} = \frac{q(x,y)}{q(x,0)\,q(y,0)} .$$

Write $v=q(y,0)$. Apply (1.4) [with radius $= r'/\sqrt{1-r^2}$] to get

$$q(g(x),g(y)) = \frac{|g(x)-g(y)|}{\sqrt{1+|g(x)|^2}\sqrt{1+|g(y)|^2}}$$

$$= \left(\frac{r}{r'}\right)^2 \frac{q(x,y)}{q(x,0)\,q(y,0)} \frac{1}{\sqrt{1+\left(\frac{r}{r'}\right)^4 \frac{1-u^2}{u^2}}} \frac{1}{\sqrt{1+\left(\frac{r}{r'}\right)^4 \frac{1-v^2}{v^2}}}$$

$$= \frac{(r\,r')^2\, q(x,y)}{\sqrt{(r')^4 u^2 + r^4(1-u^2)}\sqrt{(r')^4 v^2 + r^4(1-v^2)}} .$$

This can be written in the desired form because by the Pythagorean theorem $q(x,b)^2 + q(x,\tilde{b})^2 = 1$.

1.48. Cor. If $r = 1/\sqrt{2}$ in Thm 1.45 then g is a q-isometry.

1.49. Rmk. (1) The Ahlfors bracket $A[x,y]$ can also be written as follows

$$A[x,a]^2 = (1-|x||a|)^2 + 2(|x||a| - x\cdot a) = (1+|x||a|)^2 - 2(|x||a| - x\cdot a)$$

Therefore $1-|x||a| \le A[x,a] \le 1+|x||a|$. (see also 1.39.)

(2) It is perhaps not apparent that (1.38) implies (1.35). To check this it is expedient to start with the expression $|T_a x|^2 A[x,a]^4$ (where $T_a x$ is from 1.38) and to show that this equals $|x-a|^2 A[x,a]^2$.

2. Hyperbolic geometry

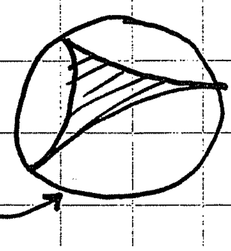
Hyperbolic geom. is synonymous with noneuclidean geom., Poincaré geom., Bolyai-Lobachevski geom.

Klein's Erlangen Program (1872). F. Klein crystallized the principles for studying geometry. Main ideas:
1. Isometries and invariants (isometries constitute a group)
2. Two configurations are equivalent if one can be mapped to the other by an isometry (element of the group)
3. The basic models of geometry are: (a) Euclidean (b) hyperbolic (c) elliptic geom.

In conclusion: basic notions of group theory are crucial.

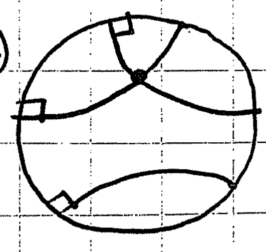
Starting points for hyperb geom. Two models, the unit ball B^n and the upper half space $H^n = R^n_+ = \{(x_1, \dots, x_n) : x_n > 0\}$.
- in these models we postulate "geometry" such that the conformal automorphisms of the model preserve length and area (cf. R^n and Euclidean isometries)
- the Parallel Postulate does not hold (the P.P. says that given a line and a point not on it "there exists one and only one straight line which passes through that point and never intersects the first line) (*)
- the sum of the angles of a triangle is $< \pi$
- "lines" are circular arcs $\perp$ to ∂B^n, ∂H^n

Triangle whose sum of angles = 0:



"Horizon"

(*)



B^n

Consider first the hyperb. geom. of H^n, later of B^n.

Def. a weight function $w : H^n \to \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ setting

(2.1) $\quad w(x) = 1/x_n\,, \quad x = (x_1, \dots, x_n) \in H^n$.

Let $\gamma : [0, \ell] \to H^n$ be a rectifiable curve, arc length as the parameter (i.e. γ is continuous and $\ell < \infty$).

We define the hyperbolic length of γ as

(2.2) $\quad \ell_h(\gamma) = \int_0^\ell |\gamma'(t)|\, w(\gamma(t))\, dt = \int_\gamma \frac{ds}{x_n}$.

If $A \subset H^n$ is (Lebesgue) measurable, then the hyperb. mea ure (volume) is defined as

(2.3) $\quad m_h(A) = \int_A w(x)^n \, dm(x)$

where m is the n-dim. Lebesgue measure. If $a, b \in H^n$, then the hyperb. distance between a and b is defined as

$$\rho(a,b) = \inf_{\alpha \in \Gamma_{ab}} \ell_h(\alpha) = \inf_{\alpha \in \Gamma_{ab}} \int_\alpha |dx|/x_n$$

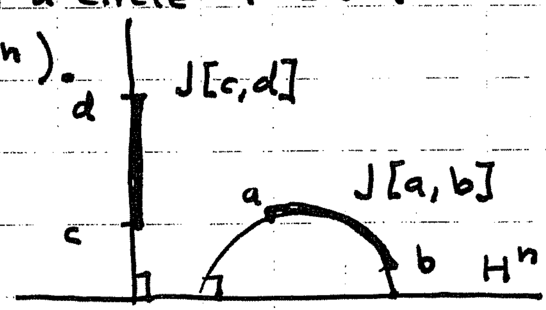
where Γ_{ab} is the set of all rectif. curves γ s.t. $a, b \in \gamma$ and $\gamma \subset \mathbb{R}^n$. It is easy to see that ρ is a metric (this is true even for a general weight function). If $A \subset H^n$, $A \neq \emptyset$, then we write $\rho(A) = \sup\{\rho(x,y) : x, y \in A\}$.

2.4. Thm (Beardon p.135) If $a, b \in H^n$, then there exists a geodesic segment $J[a,b]$ s.t.

(1) $\rho(a,b) = \int_{J[a,b]} |dx|/x_n$ $\quad$ (i.e. $J[a,b]$ is length minimizing curve)

(2) $\forall c \in J[a,b] : \quad \rho(a,b) = \rho(a,c) + \rho(c,b)$.

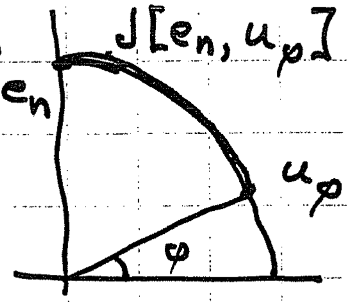
Furthermore, $J[a,b]$ is a subarc of a circle $Y \perp \partial H^n$ (a limiting case: a normal to ∂H^n).



Thm 2.4 gives for $r, s > 0$

(2.5) $\rho(re_n, se_n) = \left| \int_s^r \frac{dt}{t} \right| = \left| \log \frac{r}{s} \right|$.

For $\varphi \in (0, \pi/2)$ write $u_\varphi = (\cos\varphi)e_1 + (\sin\varphi)e_n$



Thm 2.4 $\Rightarrow$

$$(2.6)\quad \varrho(e_n, u_\varphi) = \int_{J[u_\varphi, e_n]} \frac{d\alpha}{\sin\alpha}$$

$$= \Big/_{\varphi}^{\pi/2} \log \tan\frac{\alpha}{2} = \log(1/\tan\frac{\varphi}{2}).$$

2.7. Thm. (Beardon, p.35) ϱ is $GM(H^n)$-invariant, i.e.

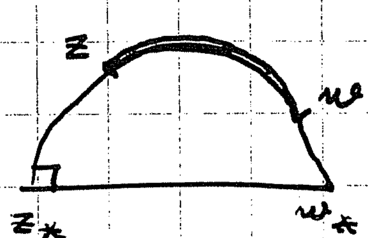
$$\varrho(x, y) = \varrho(f(x), f(y)) \quad \forall x, y \in H^n \quad \forall f \in GM(H^n)$$

and furthermore

$$\operatorname{ch}\varrho(x,y) = \cosh\varrho(x,y) = 1 + \frac{|x-y|^2}{2x_n y_n} \quad \forall x, y \in H^n.$$

Thm 2.7 contains nearly all information about the hyperb. geom. of H^n that we will use below. Note that $\varrho(x,y)$ is completely determined when the euclidean quantities $|x-y|$, x_n, y_n are known. We write $\operatorname{ch} x = \cosh x$, $\operatorname{sh} x = \sinh x$, $\operatorname{th} x = \tanh x$.

2.8. Cor. $\varrho(z, w) = \log|z_*, z, w, w_*|$ where z_*, w_* as in the figure.



Proof. By 2.7 we may assume (by making an auxiliary map $\in GM(H^n)$ if necessary), that $z_* = 0$, $w_* = \infty$ when

$$|z_*, z, w, w_*| = \frac{|z_* - w|}{|z_* - z|} > 1$$

and the proof follows from (2.5).

For $a \in H^n$ and $M > 0$ write

$$B_\varrho(a, M) = D(a, M) = \{z \in H^n : \varrho(a, z) < M\}.$$

With a reference to formula 2.7 we see, because hyperb. circles are euclidean circles, that the condition

$$\frac{|(x_1, x_2) - (0,1)|^2}{x_2} = c_1$$

defines a circle $\partial D((0,1), c_2)$ in H^2 (by 2.7 $c_2 = \operatorname{arch}(1+\frac{c_1}{2})$)
In the same way we see that $D(a,M) = B^n(z,r)$ in H^n for some z, r. Suppose now that $a = te_n$, $t > 0$. Formula (2.5) $\Rightarrow$

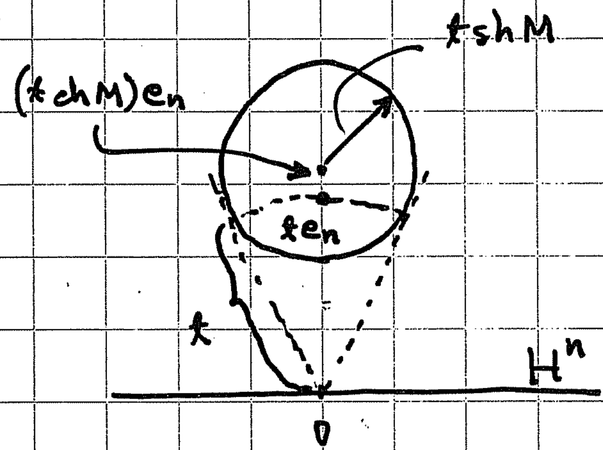
$$\alpha t e_n,\ (t/\alpha) e_n \in \partial D(te_n, M),\quad \alpha = e^M,$$

implying

$$(2.9)\quad \begin{cases} D(te_n, M) = B^n((t \operatorname{ch} M) e_n, t \operatorname{sh} M) \\ B^n(te_n, rt) \subset D(te_n, M) \subset B^n(te_n, Rt) \\ r = 1 - e^{-M}, \quad R = e^M - 1 \end{cases}$$

The numbers r and R are best possible.

Note $\begin{cases} t \operatorname{ch} M + t \operatorname{sh} M = t e^M \\ t \operatorname{ch} M - t \operatorname{sh} M = t e^{-M} \end{cases}$



Recall the inverse functions

$\operatorname{arsh} x = \log(x + \sqrt{x^2+1})$, $x \ge 0$,

$\operatorname{arch} x = \log(x + \sqrt{x^2-1})$, $x \ge 1$

$\operatorname{arth} x = \frac{1}{2} \log \frac{1+x}{1-x}$, $0 \le x < 1$.

Up to this point we have studied the hyperb. geom. in H^n. Next we consider the situation in B^n. The weight function is defined by

$$(2.10)\qquad w(x) = \frac{2}{1-|x|^2}, \quad x \in B^n.$$

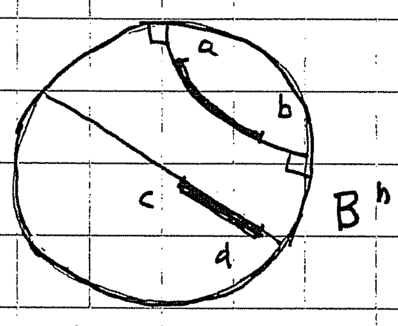
The hyperb. distance $\rho(a,b) = \rho_{B^n}(a,b)$ will be defined, with this w, in the same way as in the case of H^n. Also the <u>hyperb. volume</u> and <u>geodesic segment</u> are defined analogously.

Thms 2.4 and 2.7 have the follow. counterpart for B^n.

2.11. <u>Thm</u> (Beardon p. 40, p. 129) Given $a, b \in B^n$, there exists the <u>geodesic segment</u> $J[a,b]$ (with the same properties 1) & 2) as in Thm 2.4) which is an arc of a circle $\perp \partial B^n$ (as a limiting case: a diameter of B^n). Furthermore

$$(2.12) \quad \mathrm{sh}^2 \frac{\rho(x,y)}{2} = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}$$

and $\rho(x,y)$ is $GM(B^n)$-invariant.



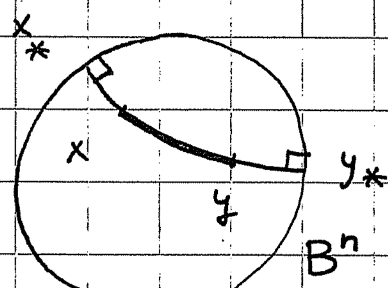
With 2.11 we may compute for $t \in (0,1)$

$$(2.13) \quad \rho(0, te_1) = \int_{[0,te_1]} \frac{2|dx|}{1-|x|^2} = \int_0^t \frac{2\,ds}{1-s^2} = \log \frac{1+t}{1-t} = 2 \,\mathrm{arth}\, t$$

This implies for $s \in (-t, t)$

$$(2.14) \quad \rho(se_1, te_1) = \log\left(\frac{1+t}{1-t} \cdot \frac{1-s}{1+s}\right).$$

Also (2.12) gives (2.14).



2.15. <u>Cor</u> $\rho(x,y) = \log |x_*, x, y, y_*|$.

<u>Proof</u>. 2.11 $\Rightarrow$ we may assume $x = 0$, $y = te_1$, $t \in (0,1)$. (2.13) $\Rightarrow$

$$\rho(x,y) = \log \frac{1+t}{1-t} = \log |x_*, x, y, y_*|. \qquad \square$$

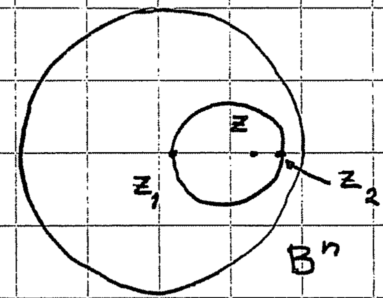
In the same way as for H^n we write

$$B_\rho(z, M) = D(z, M) = \{ y \in B^n : \rho(z,y) < M \}.$$

In the case $z = 0$ we see by (2.12) that $\partial D(0,M)$ is a sphere, hence also $T_z^{-1} \partial D(0,M) = \partial D(z,M)$ is a sphere for $z \in B^n \setminus \{0\}$ (recall that $T_z \in GM(B^n)$ maps spheres onto spheres). Therefore $D(z,M) = B^n(y,r)$ for some $y \in B^n$, $r > 0$. Let us find y and r. Let $e = z/|z|$, $z_1 = se$, $z_2 = ue$, $z_1, z_2 \in \partial D(z,M)$, $s < u$. (2.14) $\Rightarrow$

$$\rho(z_1, z) = \log \frac{1+|z|}{1-|z|}\,\frac{1-s}{1+s} = M$$

$$\rho(z_2, z) = \log \frac{1+u}{1-u}\,\frac{1-|z|}{1+|z|} = M$$



Solve these for s and u. Because $D(z,M) = B^n(y,r)$, $y = \frac{s+u}{2} e$, $r = \frac{1}{2}|u-s|$, we get

$$(2.16) \quad \begin{cases} D(z,M) = B^n(y,r) \quad \begin{cases} y = \dfrac{z(1-t^2)}{1-|z|^2 t^2}, \quad t = \operatorname{th}\dfrac{M}{2} \\ r = \dfrac{(1-|z|^2)\,t}{1-|z|^2 t^2} \end{cases} \\ \\ B^n(z, a(1-|z|)) \subset D(z,M) \subset B^n(z, A(1-|z|)) \\ a = \dfrac{t(1+|z|)}{1+|z|t}\,; \quad A = \dfrac{t(1+|z|)}{1-|z|t} \end{cases}$$

The particular case $z=0$ of (2.16) will be often used:

$$(2.17) \qquad D(0,M) = B^n(\operatorname{th}(M/2)).$$

A standard application: Let $x, y \in B^n$ and $z \in J[x,y]$ be such that $\rho(x,y) = 2\rho(x,z) = 2\rho(y,z)$. (2.17) $\Rightarrow$

$$(2.18) \quad \begin{cases} |T_x(y)| = \operatorname{th}(\rho(x,y)/2), \\ |T_z(x)| = \operatorname{th}(\rho(x,y)/4). \end{cases}$$

2.19. <u>Lemma</u> Let $x, y \in B^n$. Then $|x-y| \le 2 \operatorname{th} \frac{\rho(x,y)}{4}$.

Proof. Formula (2.16) yields (d = Eucl. diameter)

$$(*)\quad d(D(z,M)) = 2r = \frac{2(1-|z|^2)t}{1-|z|^2t^2} \le 2t = d(D(0,M)) = 2\,\mathrm{th}\frac{M}{2}.$$

Fix $x,y\in B^n$ and $z\in J[x,y]$ s.t. $\rho(x,z)=\rho(z,y)=\rho(x,y)/2$. Setting $M=\rho(x,y)/2$ in $(*)$ $\Rightarrow$

$$|x-y|\le d(\bar D(z,\rho(x,y)/2)) \le 2\,\mathrm{th}\frac{\rho(x,y)}{4}.$$

2.20. Lause. Let $f\in GM(\overline{\mathbb{R}}^n)$ with $fB^n=H^n$. Then $\rho_{H^n}(f(x),f(y))=\rho_{B^n}(x,y)$ for all $x,y\in B^n$.

Proof. Because the absolute ratio is $GM(\overline{\mathbb{R}}^n)$-invariant (1.23), the claim follows from 2.15 and 2.8.

In a cristallized form, formulas 2.7 and (2.12) contain all information about $\rho(x,y)$. Sometimes it is expedient to have upper/lower bounds for $\rho(x,y)$, instead of the exact value. For this purpose we introduce in a domain $D\subsetneq\mathbb{R}^n$

$$(2.21)\quad j_D(x,y)=\log\left(1+\frac{|x-y|}{\min\{d(x),d(y)\}}\right),\quad d(z)=d(z,\partial D)$$

For $\emptyset\ne A\subset D$ set $j_D(A)=\sup\{j_G(x,y): x,y\in A\}$. A lengthy verification shows that j_D satisfies the triangle ineq. and is a metric.

2.22. Lemma The metric j_D satisfies for all $x,y\in D$

(1) $j_D(x,y)\ge \left|\log\frac{d(x)}{d(y)}\right|$,

(2) $j_D(x,y)\le \left|\log\frac{d(x)}{d(y)}\right| + \log\left(1+\frac{|x-y|}{d(x)}\right)\le 2j_D(x,y)$.

Proof. (1) Δ-ineq $\Rightarrow d(x)+|x-y|\ge d(y)$

(2) If $d(x)\le d(y)$, the claim follows from (2.21). For $d(x)>d(y)$

$$j_D(x,y) = \log\left(1+\frac{|x-y|}{d(y)}\right) \le \log\left(\frac{d(x)}{d(y)} + \frac{d(x)}{d(y)}\frac{|x-y|}{d(x)}\right)$$ (31)

$$= \log\frac{d(x)}{d(y)} + \log\left(1+\frac{|x-y|}{d(x)}\right) \overset{(1)}{\le} 2 j_D(x,y)$$

2.23. Exer. (1) Show that $j_D(x,y) \ge \left|\log\frac{|x-w|}{|y-w|}\right|$, $w \in \partial D$.
(2) Show that $\log(1+x) \le \operatorname{arsh} x \le 2\log(1+x)$; $x \ge 0$.
(3) Show that $\operatorname{arch} x = 2\log\left(\sqrt{(x+1)/2} + \sqrt{(x-1)/2}\right)$, $x \ge 1$.
(4) Show that $2\log\left(1+\sqrt{\frac{x-1}{2}}\right) \le \operatorname{arch} x \le 2\log\left(1+\sqrt{2(x-1)}\right)$, $x \ge 1$.

2.24. Lemma (1) $j_{B^n}(x,y) \le \rho(x,y) \le 4\, j_{B^n}(x,y)$ $\forall x,y \in B^n$
(2) $j_{H^n}(x,y) \le \rho(x,y) \le 2 j_{H^n}(x,y)$, $\forall x,y \in H^n$.

Proof. (1) (2.12) $\Rightarrow$

$$\operatorname{sh}^2(\rho(x,y)/2) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)} = t^2 \le \left(\frac{|x-y|}{\min\{1-|x|,1-|y|\}}\right)^2$$

2.23(2) $\Rightarrow$ $\rho(x,y)/2 \le 2\log(1+t) \le 2 j_{B^n}(x,y)$.

The proof of the lower bound is left as an exercise

(2) Write $u = 1 + |x-y|^2/(2x_n y_n)$. 2.7 & 2.23(3) $\Rightarrow$

$$\rho_{H^n}(x,y) \le 2\log\left(1+\sqrt{2(u-1)}\right) = 2\log\left(1+\frac{|x-y|}{\sqrt{x_n y_n}}\right) \le 2 j_{H^n}(x,y)$$

The proof of the lower bound is left as an exercise.

2.25. Quasihyperbolic geometry. In a domain $D \subsetneq \mathbb{R}^n$ we can define the quasihyperbolic metric k_D, which has several properties of ρ_{B^n}. The weight function $w: D \to \mathbb{R}_+$ is defined by

(2.26) $\quad w(x) = 1/d(x,\partial D)$; $x \in D$.

As in (2.2) we define the quasihyp. length of a rectifiable curve γ