prove, hence we may assume $|y_0| > \eta^*_{K,n}(1) = c$. Let $g = f^{-1}$. Then $g B^n(|y_0|) \supset B^n(t)$ and $g(B^n(c)) \supset B^n$. Let $h(x) = x/|x|^2$ and $F = h \circ g \circ h$. Then $F \in QC_K(\mathbb{R}^n)$, $F(0) = 0$, $F(e_1) = e_1$ and $F(B^n(1/c)) \subset B^n$, $F(B^n(1/|y_0|)) \subset B^n(1/t)$. By

$$1/t = 1/|g(y_0)| = |F(y_0^*)| \le \varphi^*_{K,n}(c/|y_0|) \le \varphi_{K,n}(c/|y_0|)$$

so that $|y_0| = |f(x_0)| \le c/\varphi_{1/K,n}(1/t)$. The result follows if we take the ~~infimum~~ over all f. supre

(3) Fix $f \in QC_K(\mathbb{R}^n)$, $f(0) = 0$, $f(e_1) = e_1$ and $t \in (0,1)$. By 10.1 we have for $x \in S^{n-1}(t)$ and $y \in S^{n-1}(t)$

$$|fx|/|fe_1| = |fx| \le C_{K,n}(t)$$

$$|fy|/|fe_1| = |fy| \ge B_{K,n}(t)$$

$$\Rightarrow \quad |fx|/|fy| \le C_{K,n}(t)/B_{K,n}(t)$$

Because $\varphi_{1,n}(t) = t$, the functions $C_{1,n}(t) = \frac{t}{1-t}$ and $B_{1,n}(t) = \frac{t}{1+t}$ by Lemma 10.1. Therefore Thm 10.5(3) yields $\eta^*_{1,n}(1) \le \inf\{\frac{1+t}{1-t} : 0 < t < 1\} = 1$ which is natural. Our next goal is to deduce that for each $K > 1$ we obtain a bound for $\eta^*_{K,n}(1)$ which is explicit and $\to 1$ when $K \to 1$.

10.6. <u>Thm.</u> For $n \ge 2$ and $K > 1$ $\eta^*_{K,n}(1) \le \exp(4K(K+1)\sqrt{K-1})$.

<u>Pf.</u> By 9.3 (1) $1 - \varphi_{K,n}(\sqrt{t})^2 \ge 1 - \lambda_n^{2(1-\alpha)} t^\alpha$, $\alpha = K^{1/(1-n)}$. If $1 - \lambda_n^{2(1-\alpha)} t_0^\alpha = 1/K$, then $t_0 = (\lambda_n^{2(\alpha-1)}(K-1)/K)^\beta$ so $t_0 \le (K-1)/K$. Thus $1 - \varphi_{K,n}(\sqrt{t})^2 \ge 1/K$ for $0 < t \le t_0$. Hence by Thm 10.5(3) and Lemma 9.3

$$\eta^*_{K,n}(1) \le \frac{C_{K,n}(t_0)}{B_{K,n}(t_0)} \le K \lambda_n^{2(\beta-\alpha)} t_0^{\alpha-\beta} (1+t_0)^\beta \le K \lambda_n^{2(\beta-\alpha)} t_0^{\alpha-\beta} \left(2 - \frac{1}{K}\right)^\beta$$

$$= K^{\beta(\beta-1)}(K-1)^{1-\beta^2}\lambda_n^{2(\beta^2-1)}(2K-1)^{\beta} \equiv E$$

By elementary calculus $\max\{x^{-\sqrt{x}} : x > 0\} = e^{2/e}$ and hence

$$(K-1)^{1-K} = \left((K-1)^{-\sqrt{K-1}}\right)^{\sqrt{K-1}} \le \exp\left(\tfrac{2}{e}\sqrt{K-1}\right).$$

Consider two cases.

Case 1. $K \ge 2$. Now $(K-1)^{1-\beta^2} \le 1$ and by 9.4 ($\lambda_n^{\beta-1} \le 2^{K-1}K^K$)

$$E \le K^{K(K-1)}\lambda_n^{2(\beta^2-1)}(2K-1)^K \le K^{K(K-1)}(2^{K-1}K)^{2(K+1)}(2K-1)^K$$

$$= K^{K(K-1)}2^{2(K^2-1)}K^{2K(K+1)}(2K-1)^K$$

$$= \exp\left[(3K^2+K)\log K + K\log(2K-1) + 2(K^2-1)\log 2\right]$$

$$< \exp\left[(3K^2+K)\sqrt{K-1} + K\sqrt{2}\sqrt{K-1} + \sqrt{2}(K^2-1)\right]$$

$$< \exp\left[(3K^2+3K)\sqrt{K-1} + 2(K^2-1)\right]$$

$$= \exp\left[\sqrt{K-1}(K+1)(3K+2\sqrt{K-1})\right] \le \exp\left(4K(K+1)\sqrt{K-1}\right)$$

where we have used the inequality $\log x \le \sqrt{x-1}$, $x > 1$.

Case 2 $K \in (1,2]$. Now $(K-1)^{1-\beta} \le (K-1)^{1-K} \le \exp\left(\frac{2}{e}\sqrt{K-1}\right)$ so

$$E \le K^{K(K-1)}(2^{K-1}K^K)^{2(K+1)}(2K-1)^K \exp\left[\tfrac{2}{e}(K+1)\sqrt{K-1}\right]$$

$$= K^{K(3K+1)}4^{(K-1)(K+1)}(2K-1)^K \exp\left(\tfrac{2}{e}(K+1)\sqrt{K-1}\right)$$

$$= \exp\left[(3K^2+K)\log K + (K-1)(K+1)\log 4 + \tfrac{2}{e}(K+1)\sqrt{K-1} + K\log(2K-1)\right]$$

$$\le \exp\left[(3K^2+K)\frac{K-1}{\sqrt{K}} + \frac{2K(K-1)}{\sqrt{K}} + (K+1)(K-1)\log 4 + \tfrac{2}{e}(K+1)\sqrt{K-1}\right]$$

$$= \exp\left[3(K+1)(K-1)\sqrt{K} + (K+1)(K-1)\log 4 + \tfrac{2}{e}(K+1)\sqrt{K-1}\right]$$

$$\le \exp\left[(K+1)\sqrt{K-1}\left((3\sqrt{K}+\log 4)\sqrt{K-1} + \tfrac{2}{e}\right)\right]$$

Next we use

$$3\sqrt{K(K-1)} + (\log 4)\sqrt{K-1} + \frac{2}{e} \le \frac{3}{2}(2K-1) + \log 4 + 1 < 3K+1 < 4K$$

and hence $E \le \exp\left(4K(K+1)\sqrt{K-1}\right)$ as desired.

10.7. <u>Theorem</u>. Let $f: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-qc. Then

$$1/\eta^*_{K,n}(1/|a,b,c,d|) \le |fa,fb,fc,fd| \le \eta^*_{K,n}(|a,b,c,d|)$$

for $\forall$ quadruple of distinct points in $\overline{\mathbb{R}}^n$ where $\eta^*_{K,n}$ is as in 10.5.

<u>Pf.</u> We may assume $a=0=f(a)$, $b=e_1=f(b)$ and $d=\infty=f(d)$. Then $|a,b,c,d|=|c|$, $|fa,fb,fc,fd|=|fc|$. The upper bound follows directly from (10.2). Similarly from (10.2) we obtain

$$|fa,fb,fc,fd| = \frac{1}{|fa,fc,fb,fd|} \ge \frac{1}{\eta^*_{K,n}(|a,c,b,d|)} = \frac{1}{\eta^*_{K,n}(1/|a,b,c,d|)}.$$

10.8. <u>Cor.</u> The linear dilatation of a K-qc map $f: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, $f\infty=\infty$, satisfies $H(0,f) \le \eta^*_{K,n}(1) \le \exp(4K(K+1)\sqrt{K-1})$.

<u>Pf.</u> Fix $r>0$ and $x,y \in S^{n-1}(r)$. Now $|0,y,x,\infty| = |x|/|y| = 1$ and (because $f(\infty)=\infty$)

$$|f0,fy,fx,f\infty| = |fx-f0|/|fy-f0| \le \eta^*_{K,n}(1) \le e^{4K(K+1)\sqrt{K-1}}.$$

The claim follows from the def. of the linear dilatation

10.9. <u>Cor.</u> Let $f: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-qc map with $f\infty=\infty$ and $a,b,c \in \mathbb{R}^n$ three distinct points. Then

$$r_6(n,K)\left(\frac{|a-c|}{|a-b|+|b-c|}\right)^\beta \le \frac{|fa-fc|}{|fa-fb|+|fb-fc|} \le$$

$$r_7(n,K)\left(\frac{|a-c|}{|a-b|+|b-c|}\right)^\alpha, \quad \alpha = K^{1/(1-n)} = 1/\beta,$$

where r_6, r_7 depend only on n, K and equal to 1 for $K=1$.