Roughly speaking Thm 8.23 says that the greater the set $\overline{\mathbb{R}}^n \setminus fB^n$ is, the smaller $q(f(x), f(y))$ is (when $n, K, \rho(x,y)$ are fixed). I.e. the larger set f omits, the less it can oscillate.

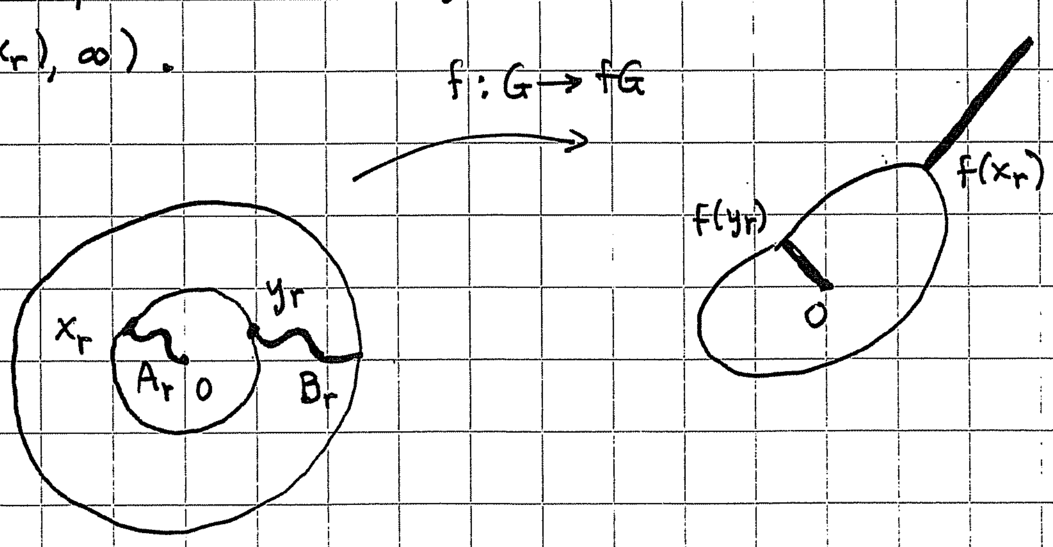
The <u>linear</u> <u>dilatation</u> of $f: G \to \mathbb{R}^n$ at $x \in G$ is defined by

$$(8.24) \begin{cases} H(x,f) = \limsup_{r \to 0} \dfrac{L(x,f,r)}{\ell(x,f,r)} \\ L(x,f,r) = \max\{|f(x) - f(z)| : |x-z| = r\} \\ \ell(x,f,r) = \min\{ \qquad \qquad \} \end{cases}$$

8.25. <u>Thm</u> If $f: G \to \mathbb{R}^n$ is non-const. qr and $x \in G$ then

$$H(x,f) \le c(n, K_0(f) i(x,f)) < \infty .$$

<u>Pf.</u> May ass. $x = 0 = f(x)$. Let σ_0 be as in 7.19, $U = U(0,f,\sigma_0)$ and choose $t > 0$ s.t. $B^n(3t) \subset U$. For each $r \in (0,t]$ choose $x_r, y_r \in S^{n-1}(r)$ s.t. $|f(x_r)| = L(0,f,r)$, $|f(y_r)| = \ell(0,f,r)$. Let A_r be the y_r-component of $f^{-1}[0, f(y_r)]$ and B_r the x_r-component of $f^{-1}[f(x_r), \infty)$.



Then $0 \in A_r$ and $B_r \cap \partial U \ne \emptyset$ by 7.25. Let $\Gamma_r = \Delta(A_r, B_r; U)$. Then

$$M(\Gamma_r) + \omega_{n-1}\left(\log \frac{3t}{r}\right)^{1-n} \ge M(\Delta(A_r, U \cap B_r; \mathbb{R}^n)).$$

Symmetrization yields

$$M(\Delta(A_r, U \cap B_r; \mathbb{R}^n)) \geq M(\Delta([0, re_1], [-re_1, -3t e_1]))$$
$$= \tau((3t+r)/(3t-r)).$$

If $|f(x_r)| > |f(y_r)|$, then

$$M(f\Gamma_r) \leq \tau\left(\frac{|f(x_r)|}{|f(y_r)|} - 1\right).$$

This holds trivially if $|f(x_r)| = |f(y_r)|$ (setting $\tau(0) = \infty$).

On the other hand

$$M(\Gamma_r) \leq K_0(f)\, i(x,f)\, M(f\Gamma_r)$$

which together with the previous relations yields

$$\tau\left(\frac{3t+r}{3t-r}\right) - \omega_{n-1}\left(\log\frac{3t}{r}\right)^{1-n} \leq K_0(f)\, i(x,f)\, \tau\left(\frac{|f(x_r)|}{|f(y_r)|} - 1\right)$$

Letting $r \to 0$ yields

$$H(0,f) \leq 1 + \tau^{-1}(\tau(1)/(K_0(f)\, i(x,f))) = c(n, K_0(f) i(x,f)).$$

8.26. <u>Cor.</u> If $f: \mathbb{R}^n \to \mathbb{R}^n$ is qc and $f(0)=0$, then for $x, y \in \mathbb{R}^n$, $|x| = |y|$

$$|f(x)| \leq c(n, K_0(f))\, |f(y)|.$$

Recall the foll. estimate from 5.25 (2)

(8.27) $$2^{n-1} c_n \log\frac{s+1}{s-1} \leq \gamma_n(s) \leq 2^{n-1} c_n\, \mu\left(\frac{s-1}{s+1}\right).$$

We will next deduce a <u>dimension cancellation</u> lemma.

For $n \geq 2$, $K \in (0, \infty)$; $0 < r < 1$ write

(8.28) $$\varphi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(\tfrac{1}{r})).$$

8.29. <u>Lemma</u>. Let $K \in (0,\infty)$, $n \geq 2$, $r \in (0,1)$. Then

$$\frac{1 - \mu^{-1}(K \log\frac{1+r}{1-r})}{1 + \mu^{-1}(K\log\frac{1+r}{1-r})} \leq \varphi_{K,n}(r) \leq \operatorname{th}\left(\frac{K}{2}\mu\left(\frac{1-r}{1+r}\right)\right).$$

Pf. Set $a = 2^{n-1} c_n$, $u = a \log \frac{1+1/s}{1-1/s} = 2a \,\mathrm{arth}(1/s)$, $v = a\,\mu\left(\frac{s-1}{s+1}\right)$. (8.28) $\Rightarrow$

(8.30) $$\gamma_n^{-1}(u) \geq s = 1/\mathrm{th}\,(u/(2c))$$

$$\gamma_n^{-1}(v) \leq s = \frac{1+b}{1-b} \; ; \; b = \mu^{-1}(v/c)$$

Because γ_n is decr., also γ_n^{-1} is. Substitute (8.27) in (8.30) to get

$$\gamma_n^{-1}(K\gamma_n(s)) \geq \gamma_n^{-1}\left(K a\, \mu\left(\frac{s-1}{s+1}\right)\right) \geq \frac{1}{\mathrm{th}\left(\frac{K}{2}\mu\left(\frac{s-1}{s+1}\right)\right)}$$

On the other hand

$$\gamma_n^{-1}(K\gamma_n(s)) \leq \gamma_n^{-1}\left(K a \log\frac{s+1}{s-1}\right) \leq \frac{1+d}{1-d} \; ; \; d = \mu^{-1}\left(K \log\frac{s+1}{s-1}\right).$$

N.B: The bounds in 8.29 are independent of n! However, one can prove that $2^{n-1}c_n \to 0$ when $n \to \infty$ and hence $\gamma_n(s) \to 0$ when $n \to \infty$, see (8.27). Therefore 8.29 shows that dimension cancellation occurs.

8.31. Cor. The upper bound for $H(x,f)$ in 8.25 has a majorant independent of n.

Pf. Because $\gamma_n(s) = 2^{n-1} \cdot \tau_n(s^2-1)$, we get for $A > 0$

$$x = 1 + \tau^{-1}(A\tau(s)) \iff \tau(x-1)\; A\tau(s)$$

$$\iff \gamma(\sqrt{x}) = A\gamma(\sqrt{1+s})$$

$$\iff x = \left(\gamma^{-1}(A\gamma(\sqrt{1+s}))\right)^2$$

$$\iff (*) \quad x = 1/\varphi_{A,n}(1/\sqrt{1+s})^2$$

8.25 $\Rightarrow$

$$H(x,f) \leq 1 + \tau^{-1}\left(\frac{\tau(1)}{K_0\, i(x,f)}\right) \overset{(*)}{=} 1/\varphi_{K_0(f)i(x,f),n}(1/\sqrt{2})^2$$

8.29 $\Rightarrow$ claim.

9. Distortion theory

__Basic problem:__ Study change of distances under qr maps.
__Basic results:__ Qr maps (a) are Hölder-continuous (b) satisfy a generalized Schwarz lemma.

	(a)	(b)
$n=2$	Ahlfors [A4] -54	Hersch-Pfluger [HEP] -52 Belinskii - 53
$n \ge 3$	Callender [CA] -60 Gehring [G2] - 62 Reshetnyak [R1] - 66	Martio-Rickman-Väisälä [MRV1] -69

The aforementioned n-dim. results depend essentially on n: when $n \to \infty$ the constants may tend to 0 or ∞. Dimension-free can be found in [AVV1], [AVV2].

First we will sharpen Lemma 8.29. Define $M_n(r)$ by

$$(9.1) \qquad \gamma_n(s) = \omega_{n-1} / M_n(1/s)^{n-1}$$

Then by (3.23) $M_2(r) = \mu(r)$. It is easy to show that

$$(9.2) \qquad \varphi_{K,n}(r) = M_n^{-1}(\alpha M_n(r)), \quad \alpha = K^{1/(1-n)}.$$

9.3. __Lemma.__ For $n \ge 2$ and $r \in (0,1)$ we have

$$(1) \qquad \varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^{\alpha}, \quad \text{for } K \ge 1, \ \alpha = K^{1/(1-n)},$$

$$(2) \qquad \varphi_{1/K,n}(r) \ge \lambda_n^{1-\beta} r^{\beta}, \quad \text{for } K \ge 1, \ \beta = 1/\alpha .$$

__Pf.__ (1) By the pf of Lemma 5.21 we see that $M(r) + \log r$ is decreasing on $(0,1)$. Set $s = \varphi_{K,n}(r) \ge r$. Then

$$M(r) + \log r \ge M(s) + \log s$$

$$M(r) + \log \frac{r}{\lambda_n} \ge M(s) + \log \frac{s}{\lambda_n} \overset{(9.2)}{=} \alpha M(r) + \log \frac{s}{\lambda_n}$$

Note that by (5.22) $\log \lambda_n \ge M(r) + \log r \Rightarrow 0 \le -M(r) + \log \frac{\lambda_n}{r}$.

This yields

$$-\alpha M(r) + \alpha \log \frac{\lambda_n}{r} \le -\alpha M(r) + \log \frac{\lambda_n}{s}$$

$$\Longleftrightarrow \alpha \log \frac{\lambda_n}{r} \le \log \frac{\lambda_n}{s} \Longleftrightarrow s \le \lambda_n^{1-\alpha} r^{\alpha}.$$

The part (2) is proved in the same way.

9.4. <u>Cor.</u> For $K \ge 1$ we have

$$\varphi_{K,n}(r) \le 2^{1-1/K} K r^{\alpha}, \quad \alpha = K^{1/(1-n)}.$$

<u>Pf.</u> $5.23 \Rightarrow \log \lambda_n \le n-1+\log 2 \;\; \forall n \ge 2$. Clearly $1-\alpha \le 1-1/K$ and

$$(1-\alpha)(n-1) = (1-K^{1/(1-n)})(n-1) \le \log K$$

where in the last step the inequality $1-e^{-x} \le x$, $x > -1$, was used. Now

$$(1-\alpha)\log \lambda_n \le (n-1+\log 2)(1-\alpha) \le \log K + (1-1/K)\log 2.$$

This last inequality together with 9.3 yields

$$\varphi_{K,n}(r) \le \lambda_n^{1-\alpha} r^{\alpha} \le 2^{1-1/K} K r^{\alpha}.$$

After these preliminary steps we may prove Schwarz' lemma for qr maps.

9.5. <u>Thm.</u> Let $f: B^n \to B^n$ be K-qr and $\alpha = K^{1/(1-n)}$. Then for all $x, y \in B^n$

(1) $$\operatorname{th} \frac{\rho(f(x), f(y))}{2} \le \varphi_{K,n}\left(\operatorname{th} \frac{\rho(x,y)}{2}\right) \le \lambda_n^{1-\alpha}\left(\operatorname{th} \frac{\rho(x,y)}{2}\right)^{\alpha},$$

(2) $$\rho(f(x), f(y)) \le K \mu(e^{-\rho(x,y)}) \le K(\rho(x,y) + \log 4).$$

<u>Pf.</u> Fix $x, y \in B^n$. Because $fB^n \subset B^n$, 6.6 and 6.7 (1) $\Rightarrow$

$$\mu_{fB^n}(fx, fy) \ge \mu_{B^n}(fx, fy) = \gamma_n(1/\operatorname{th} b)$$

where $b = \rho(fx, fy)/2$. 8.21(1) & 6.7(1) $\Rightarrow$

$$\mu_{fB^n}(fx, fy) \le K \mu_{B^n}(x, y) = K \gamma_n(1/\operatorname{th} a)$$

where $a = \rho(x,y)/2$. 9.3 (1) $\Rightarrow$ (1).

Part (2) follows from the above inequalities (cf. 5.33 and (5.32))

$$2^{n-1}c_n\,\varrho(fx,fy) \le \gamma_n(1/\text{th}\, b)$$

$$\gamma_n(1/\text{th}\, a) \le 2^{n-1}c_n\,\mu(e^{-\varrho(x,y)}) \le 2^{n-1}c_n\,(\varrho(x,y)+\log 4).$$

9.6. Cor. Let $f: B^n \to B^n$ K-qc, $f(0)=0$ and $fB^n = B^n$. Then

$$\varphi_{1/K,n}(|x|) \le |f(x)| \le \varphi_{K,n}(|x|).$$

Pf. $(2.13) \Rightarrow \text{th}\,\dfrac{\varrho(0,f(x))}{2} = |f(x)|$.

The second ineq. follows from 9.5 (1). Because $fB^n = B^n$ and f is injection we may apply 9.5 (1) also to the K-qc map $f^{-1}: B^n \to B^n$, obtaining

$$|x| = |f^{-1}(f(x))| \le \varphi_{K,n}(|f(x)|)$$

Observing that $\varphi_{K,n}^{-1} = \varphi_{1/K,n}$ this yields the first inequality.

9.7. Rmk. (1) The K-qc map $x \mapsto |x|^{\alpha-1}x$, $\alpha = K^{1/(1-n)}$, (cf. 8.17 (2)) shows that the exponent in 9.5 and 9.6 is sharp.

(2) For $n=2$, $K=1$, Thm 8.5 implies that the Poincaré distance ϱ does not grow under anal. functions $f: B^n \to B^n$.

(3) In [LV, pp. 60-65] it is shown that for $n=2$ the bound in 9.6 is attained for each $K \ge 1$.

Thm 9.5 shows that a K-qc map $f: B^n \to B^n$ is unif. cont. as a map $f: (B^n,\varrho) \to (B^n,\varrho)$, with the modulus of continuity

(9.8) $$\omega_f(t) = 2\,\text{arth}\,\varphi_{K,n}(\text{th}\,\tfrac{t}{2}).$$

In the case $fB^n \ne B^n$ it is natural to expect that here the target space (B^n,ϱ) could be replaced by (fB^n, k_{fB^n}).

9.9. <u>Ex.</u> We show that the anal. function $f: B^2 \to B^2 \setminus \{0\} = fB^2$, $f(z) = \exp\left(\frac{z+1}{z-1}\right)$, $z \in B^2$, is not unif. cont. as a map $f: (B^2, \varrho) \to (fB^2, k_{fB^2})$. Let $x_j = (e^j - 1)/(e^j + 1)$, $j = 1, 2, \ldots$ Then (2.13) $\Rightarrow$ $\varrho(0, x_j) = j$ and thus $\varrho(x_j, x_{j+1}) = 1$. Write $Y = B^2 \setminus \{0\}$. Because $f(x_j) = \exp(-e^j)$ we obtain by 2.30

$$k_Y(fx_j, fx_{j+1}) \geq j_Y(fx_j, fx_{j+1}) = \log\left(1 + (\exp e^{j+1})(\exp(-e^j) - \exp(-e^{j+1}))\right)$$
$$= e^{j+1} - e^j \to \infty, \quad j \to \infty.$$

Because $\varrho(x_j, x_{j+1}) = 1$, we see that $f: (B^2, \varrho) \to (Y, k_Y)$ is not unif. cont.

In Ex 9.9 $\partial fB^2 = \{0\} \cup \partial B^2$. Consider next the case when ∂fB^2 does not have point components.

9.10. <u>Thm</u> Let $f: B^n \to \mathbb{R}^n$ be a qr map and $G \subset \mathbb{R}^n$ a domain such that ∂G is connected and $fB^n \subset G$.

(1) Then $f: (B^n, \varrho) \to (G, j_G)$ is unif. cont.

(2) If G is uniform, then $f: (B^n, \varrho) \to (G, k_G)$ is unif. cont.

<u>Pf.</u> (1) As in 9.5, apply also 6.30.

(2) G is unif. $\Leftrightarrow$ $\exists$: $k_G(x,y) \leq c\, j_G(x,y)$ $\forall x, y \in G$. The pf follows now from (1).

9.11. <u>Lemma</u> Let $G, G' \subsetneq \mathbb{R}^n$ be domains, let G be uniform, and let $\partial G'$ be connected. If $f: G \to \mathbb{R}^n$ is qr and $fG \subset G'$ then $j_{G'}(fx, fy) \leq a_1 j_G(x,y) + a_2$ $\forall x, y \in G$. Here a_1, a_2 are posit. numbers that only depend on n, $K(f)$ and the const. in the def. of a uniform domain.

Pf. Claim. For a domain $G \subset \mathbb{R}^n$ we have $\mu_G(x,y) \le b_1 k_G(x,y) + b_2$ where b_1, b_2 only depend on n.

Pf of claim. Fix $x, y \in G$ and choose $a_1 = x$, $a_j \in J_G[x,y]$ $a_{m+1} = y$ s.t. $|a_j - a_{j+1}| = \frac{1}{2} d(a_j)$, $j = 1, \ldots, m-1$, $|a_m - a_{m-1}| < \frac{1}{2} d(a_m)$

Then $j = 1, \ldots, m-1$

$$\log(1 + \tfrac{1}{2}) \le j_G(a_j, a_{j+1}) \le k_G(a_j, a_{j+1})$$

and therefore

$$(m-1) \log \frac{3}{2} \le \sum_{j=1}^{m-1} k_G(a_j, a_{j+1}) \le k_G(x,y)$$

implying $m \le 1 + k_G(x,y)/\log\frac{3}{2}$. Now for $j = 1, \ldots, m$

$$M_j = M(\Delta([a_j, a_{j+1}], \partial G)) \le \gamma(2)$$

and

$$\mu_G(x,y) \le \sum_{j=1}^{m} M_j \le m\,\gamma(2) \le \gamma(2) + \gamma(2)\, k_G(x,y)/\log\frac{3}{2}.$$

We can choose $b_2 = \gamma(2)$ and $b_1 = b_2/\log\frac{3}{2}$. □

By this claim and 6.30

$$c_n\, j_{G'}(f(x), f(y)) \le \mu_{G'}(f(x), f(y)) \le K_I(f)\, \mu_G(x,y)$$

$$\le K_I(f)(b_1 k_G(x,y) + b_2) \le K_I(f)\, b_1\, j_G(x,y) + K_I(f)\, b_2.$$

9.12. Thm. Let $f: B^n \to B^n$ be K-qr, $N = N(f, B^n) < \infty$, $x, y \in B^n$, $f(x) \ne f(y)$, $\varrho = \varrho(x,y)$, $\varrho' = \varrho(fx, fy)$. Then

(1) $$\mathrm{sh}^2 \frac{\varrho'}{2} \le \tau^{-1}\left(\frac{1}{NK}\tau\left(\mathrm{sh}^2 \frac{\varrho}{2}\right)\right)$$

(2) $$\mathrm{th}\,\frac{\varrho'}{4} \le 2\left(\mathrm{th}\,\frac{\varrho}{4}\right)^{1/(NK)}$$

Pf. $6.7 \Rightarrow \lambda_{B^n}(x,y) = \frac{1}{2}\tau(\mathrm{sh}^2 \frac{\varrho}{2})$

The pf of (1) follows from this 6.6 and 8.21 (2)

For the pf of (2) we need 8.21(2) and the inequalities (cf. 6.8)

$$\tau(\mathrm{sh}^2 \tfrac{\varrho}{2}) \ge -c_n \log \mathrm{th}\,\tfrac{\varrho}{4}\;; \quad \tau(\mathrm{sh}^2 \tfrac{\varrho'}{2}) \le c_n \log(2/\mathrm{th}\,\tfrac{\varrho'}{4}).$$

9.13. Thm. Let $f: B^n \to B^n$ be K-qc and $f(0)=0$. Then $\forall x \in B^n$

$$A(|f(x)|) \le 2\, A(|x|)^{1/K} \quad ; \quad A(t) = t/(1+\sqrt{1-t^2}).$$

Pf. Use 9.12(2) and

$$th\left(\tfrac{1}{4}\log\tfrac{1+s}{1-s}\right) = th\left(\tfrac{1}{2}\,\mathrm{arth}\, s\right) = A(s).$$

9.14. Rmk. Putting together 9.3, 9.6, and 9.13 one can prove:
If $f: B^n \to B^n = fB^n$, $f(0)=0$ is K-qc then

$$4^{1-K^2}|x|^K \le |f(x)| \le 4^{1-1/K^2}|x|^{1/K}$$

9.15. Thm. Let $f: B^n \to B^n$ be K-qr. Then

$$|f(x)-f(y)| \le b_K\left(th\,\frac{\varrho(x,y)}{2}\right) \qquad \forall\, x,y \in B^n$$

where $b_K(s) = 2\varphi_{K,n}(s)\big/\left(1+\sqrt{1-\varphi_{K,n}(s)^2}\right)$.

Pf. Write $t' = \frac{1}{2}\varrho(fx,fy)$. 2.19 $\Rightarrow$

$$|fx-fy| \le 2\,th\,\frac{t'}{2} = \frac{2\,th\,t'}{1+\sqrt{1-th^2 t'}}$$

9.5(1) $\Rightarrow$ $\quad th\,t' \le \varphi_{K,n}\left(th\,\tfrac{1}{2}\varrho(x,y)\right)$.

9.16. Thm. For $n\ge 2$, $r\in(0,1)$, $K\in[1,\infty)$ there exists $a(r)$ s.t. $\lim_{r\to 0} a(r) = 1$ and if $f: B^n\to B^n$ is K-qr then $\forall\, x,y \in B^n$

$$|fx-fy| \le a(r)\,\lambda_n^{1-\alpha}|x-y|^\alpha \le a(r)\,2^{1-1/K}K\,|x-y|^\alpha; \quad \alpha = K^{1/(1-n)}.$$

Pf. (2.12) $\Rightarrow$

$$(9.17) \qquad th^2\,\frac{\varrho(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)}$$

For $x,y \in B^n(r)$ (9.17) $\Rightarrow$ $th\,\dfrac{\varrho(x,y)}{2} \le th\,\dfrac{\varrho(-re_1,re_1)}{2} \le \dfrac{2r}{1+r^2}$

9.15, 9.5 $\Rightarrow$

$$|fx-fy| \le b_K\left(th\,\frac{\varrho}{2}\right) \le \frac{2\lambda_n^{1-\alpha}\left(th\,\frac{\varrho(x,y)}{2}\right)^\alpha}{1+\sqrt{1-\varphi_{K,n}^2\left(\frac{2r}{1+r^2}\right)}}$$

$$\overset{(*)}{\le} \left(1+\varphi_{K,n}\left(\frac{2r}{1+r^2}\right)\right)\frac{\lambda_n^{1-\alpha}\,|x-y|^\alpha}{\left[|x-y|^2+(1-|x|^2)(1-|y|^2)\right]^{\alpha/2}} \le \lambda_n^{1-\alpha}\,a(r)\,|x-y|^\alpha$$

where $a(r) = \left(1+\varphi_{K,n}\left(\frac{2r}{1+r^2}\right)\right)(1-r^2)^{-\alpha}$. Above we used

(*) $\quad 2/(1+\sqrt{1-x^2}) \le 1+x\,, \quad 0 \le x \le 1.$

9.18. Thm. Let $f : B^n \to \mathbb{R}^n \setminus \{0\}$ be a qr map with $N = N(f, B^n) < \infty$. Then for $x, y \in B^n$

$$|f(x)| \le |f(y)|\left(1+\tau^{-1}\left(A\,\tau\left(\mathrm{sh}^2\frac{\varrho(x,y)}{2}\right)\right)\right), \quad A = 1/(N K_0(f)).$$

Pf. If $|f(x)| \le |f(y)|$ there is nothing to prove. Therefore we may assume $|f(x)| > |f(y)|$. Let $G = \mathbb{R}^n \setminus \{0\}$. By 3.30 & 6.6

$$\lambda_{fB^n}(fx, fy) \le \lambda_G(fx, fy) \le M(\Delta([0, f(y)], [f(x), \infty))) \le \tau\left(\frac{|f(x)|}{|f(y)|} - 1\right).$$

By 6.7 $\lambda_{B^n}(x,y) = \frac{1}{2}\tau\left(\mathrm{sh}^2\frac{\varrho(x,y)}{2}\right)$. Applying 8.21(2) together with the above ineq. yields the assertion.

9.19. Thm. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-qc map with $f(0) = 0$. Then

$$\frac{|f(x) - f(y)|}{\min\{|f(x)|, |f(y)|\}} \le \tau^{-1}\left(\frac{1}{K\sqrt{2}}\,\tau\left(\frac{|x-y|}{\min\{|x|, |y|\}}\right)\right) \quad \forall\, x, y \in \mathbb{R}^n \setminus \{0\}$$

Pf. Let $D = \mathbb{R}^n \setminus \{0\}$, $D' = \mathbb{R}^n \setminus \{0\}$. Then $fD = D'$ and $\forall x, y \in D$

$$\lambda_D(x,y) \ge \tau(r_D(x,y)) \quad \left[\text{because } \lambda_D(x,y) \ge \tau\left(\frac{|x-y|}{\min\{|x|,|y|\}}\right)\right]$$

$$\lambda_{D'}(fx, fy) \le \sqrt{2}\,\tau(r_{D'}(fx, fy)) \quad [\text{by } 6.22]$$

$$\lambda_D(x,y) \le K\,\lambda_{D'}(fx, fy)$$

These imply the assertion

$$r_{D'}(fx, fy) \le \tau^{-1}\left(\frac{1}{K\sqrt{2}}\,\tau(r_D(x,y))\right).$$

9.20. Thm. Let $G \subsetneq \mathbb{R}^n$ be c-QED and $f: G \to fG \subset \mathbb{R}^n$ be K-qc. Then for all $x, y \in G$, $f(x) \neq f(y)$

$$r_{fG}(fx, fy) \le \tau^{-1}(d\,\tau(r_G(x,y))), \quad d = \frac{c}{2^n K}.$$

Pf. Write $r = r_G(x,y)$

$$\lambda_G(x,y) \ge c\,\tau(4r^2+4r) \ge 2^{1-n} c\,\tau(r)$$

By

$$\lambda_{fG}(fx, fy) \le \sqrt{2}\,\tau(r_{fG}(fx, fy))$$

These ineq. together with imply the assertion.

9.21. Ex. We show that the c-QED hypothesis in 9.20 cannot be left out. Let $G = B^2 \setminus [0,1)$ and $f: G \to fG = B^2 \cap H^2$ be the konf. map $f(z) = \sqrt{z}$, $z \in G$. Write $x_j = (1/2, 1/j)$, $y_j = (1/2, -1/j)$ $j = 4, 5, \ldots$ Then $r_G(x_j, y_j) = 2$ but it is easy to see that $r_{fG}(f(z_j), f(y_j)) \to \infty$, $j \to \infty$. (Note: It is easy to show that G is not c-QED for any $c > 0$.)

9.22. Rmk. Let $f: B^n \to B^n$, $f(0) = 0$, be K-qc. Then $\mathrm{sh}^2(\varrho(x,0)/2) = |x|^2/(1-|x|^2) = A(|x|)$ and 9.12 (1) yields

$$(*) \qquad A(|f(x)|) \le \tau^{-1}(\tau(A(|x|))/K)$$

Because $2^{n-1}\tau(s) = \gamma(\sqrt{1+s})$ we see that $2^{n-1}\tau(A(t)) = \gamma(1/\sqrt{1-t^2})$.

Therefore $(*) \Leftrightarrow$

$$\gamma(1/\sqrt{1-|fx|^2}) \ge \gamma(1/\sqrt{1-|x|^2})\frac{1}{K} \quad \Rightarrow$$

$$\Rightarrow \quad 1/(1-|fx|^2) \le \gamma^{-1}(\gamma(1/\sqrt{1-|x|^2})/K)^2 = \varphi_{1/K,n}(\sqrt{1-|x|^2})^{-2}$$

$$\Rightarrow (**) \quad |f(x)|^2 \le 1 - \varphi_{1/K,n}(\sqrt{1-|x|^2})^2$$

For $n = 2$ $1 - \varphi_{1/K,2}(\sqrt{1-r^2})^2 \equiv \varphi_{K,2}(r)^2$, hence $(**)$ is the same as the bound 9.6. For $n \ge 3$ $(**)$ improves the Schwarz lemma.

10. Quadruples and qc maps

The absolute ratio of a quadruple of points in $\overline{\mathbb{R}}^n$ is invariant under Möbius transformations. We will prove here a similar result for K-qc maps, which for $K \to 1$ yields the property of Möbius transformations.

10.1. <u>Lemma</u>. Let $f: \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-qc with $f(0)=0$, $f(\infty)=\infty$. Then for $0<|z|<|x|$, $x, z \in \mathbb{R}^n$,

$$B_{K,n}(|z|/|x|) \le |f(z)|/|f(x)| \le C_{K,n}(|z|/|x|)$$

where $C_{K,n}(t) = \frac{u}{1-u}$; $u = \varphi_{K,n}(\sqrt{t})^2$, $B_{K,n}(t) = \varphi_{1/K,n}\left(\sqrt{\frac{t}{1+t}}\right)^2$ and $B_{K,n}(t) = C^{-1}_{K,n}(t)$.

<u>Pf.</u> We prove only the upper bound for $|f(z)|/|f(x)|$, since the lower bound is proved in the same way. Let $\Delta = \Delta([0,z],[x,\infty])$. Then by 3.30

$$M(\Delta) \le \tau(|x|/|z|-1) = 2^{1-n}\gamma\left(\sqrt{|x|/|z|}\right)$$

while the spherical symm. yields

$$M(f\Delta) \ge \tau_n(|f(x)|/|f(z)|) = 2^{1-n}\gamma\left(\sqrt{1+|f(x)|/|f(z)|}\right)$$

Because $M(f\Delta) \le K M(\Delta)$ these ineq's give

$$\gamma\left(\sqrt{1+|f(x)|/|f(z)|}\right) \le K\gamma\left(\sqrt{|x|/|z|}\right).$$

Since γ is decr. the desired bound follows from the def. of $\varphi_{K,n}$.

For $n \ge 2$, $K \ge 1$, let $QC_K(\mathbb{R}^n)$ be the class of all K-qc maps $\mathbb{R}^n \to \mathbb{R}^n$, and for $t \in (0,\infty)$ let

(10.2) $\quad \eta^*_{K,n}(t) = \sup\{|fx| : |x| \le t,\ f \in QC_K(\mathbb{R}^n),\ f(0)=0,\ f(e_1)=e_1\}.$

10.3. **Lemma.** The foll. holds for $n \geq 2$, $K \geq 1$

(1) $\eta^*_{K,n}(1) = \sup\{|fx|/|fy| : |x|=|y|>0,\ f \in QC_K(\mathbb{R}^n),\ f(0)=0\}$

(2) $\eta^*_{K,n}(1) = \sup\{|fx|/|fy| : |x|=|y|=t,$ —"— $\}$

for each $t>0$.

Pf. First, let a stand for the RHS of (1). It is clear that $\eta^*_{K,n}(1) \leq a$. Fix $b \in (0,a)$ and choose $f \in QC_K(\mathbb{R}^n)$ with $f(0)=0$ and $x_0, y_0 \in \mathbb{R}^n \setminus \{0\}$ with $|x_0|=|y_0|$ and $|f(x_0)|/|f(y_0)| > b$. Let $h_j : \mathbb{R}^n \to \mathbb{R}^n$ be similarities with $h_j(0)=0$, $j=1,2$, $h_1(e_1)=y_0$, $h_2(f(y_0))=e_1$. Then $g = h_2 \circ f \circ h_1 \in QC_K(\mathbb{R}^n)$ satisfies $g(0)=0$, $g(e_1)=e_1$ and

$$|g(h_1^{-1}(x_0))| = |h_2(f(x_0))| = |f(x_0)|/|f(y_0)| > b$$

Since $b \in (0,a)$ was arbitrary we have proved that $\eta^*_{K,n}(1) \geq a$, and (1) follows. The proof for (2) is similar.

For $n \geq 2$, $K \geq 1$, $r \in (0,1)$ let

(10.4) $$\varphi^*_{K,n}(r) = \sup\{|fx| : |x|=r,\ f(0)=0,\ f: B^n \to B^n \text{ is } K\text{-qc}\}$$

10.5. **Thm.** The foll. holds for $n \geq 2$, $K \geq 1$:

(1) $\eta^*_{K,n}(t) \leq \eta^*_{K,n}(1)\, \varphi^*_{K,n}(t)$, $0 \leq t \leq 1$,

(2) $\eta^*_{K,n}(t) \leq \eta^*_{K,n}(1) / \varphi_{1/K,n}(1/t)$, $t>1$,

(3) $\eta^*_{K,n}(1) \leq \inf\limits_{0<t<1} C_{K,n}(t)/B_{K,n}(t)$.

Pf. (1) Let $f \in QC_K(\mathbb{R}^n)$, $f(0)=0$, $f(e_1)=e_1$. Set $g(x) = f(x)/\eta^*_{K,n}(1)$. then $g(B^n) \subset B^n$ and by (10.4) $|g(x)| \leq \varphi^*_{K,n}(|x|)$, $x \in B^n$.

(2) Fix $t>1$, $|x_0|=t$, $y_0 = f(x_0)$ so that $|y_0| = |f(x_0)| = \max\{|fx| : |x|=t\}$. If $|y_0| \leq \eta^*_{K,n}(1)$ there is nothing to