7.30. The winding map. Let (r, φ, z), $r \geq 0$, $\varphi \in [0, 2\pi)$, $z \in \mathbb{R}$, be the cylindrical coordinates in $\mathbb{R}^3$. Fix an integer $p > 1$ and consider the map $f(r, \varphi, z) = (r, p\varphi, z)$. Then f is discrete open and $N(f, B^3) = p$. In fact, the z-axis $\{(r, \varphi, z) \in \mathbb{R}^3 : r = 0\}$ is B_f and $fB_f = B_f$ and $N(y, f, B^3) = p$ for every $y \in B^3 \setminus (fB_f)$. In some respects, f resembles the power $z \mapsto z^p$.

7.31. Simply connected domain. Recall that a domain $D \subset \mathbb{R}^n$ is simply connected iff every path $\alpha : [0,1] \to D$ with $\alpha(0) = \alpha(1)$ can be continuously deformed to the point $\alpha(0)$ in D.

Clearly $B^2 \setminus \{0\}$ is not simply connected. Observe that $B^n \setminus \{0\}$ is simply connected if $n \geq 3$.

7.32. Topological fact (monodromy theorem, covering map) Let $f : G \to fG$, $G, fG \subset \mathbb{R}^n$, be a closed local homeomorphism and fG be simply connected. Then f is 1-to-1, i.e. injective.

7.33. Open problem. Let $f : B^n \to fB^n$ be discrete, open, closed, $n \geq 3$, and let B_f be compact. Is it true that f is 1-to-1 ? (I first found this problem myself, working on my PhD thesis ca 1975 and it is one of the open problems on p. 193 of CGQM.)

8. The definition of a qr map

History
1928 Grötzsch defines qc and qr maps, $n=2$
1938 Lavrentiev considers the case $n=3$
1938–44 Teichmüller
1950 Beurling-Ahlfors: Extremal length
1961 Gehring, Väisälä: system. study of qc, $n \ge 3$
1966–69 Reshetnyak builds qr theory, $n \ge 3$
1969–72 Martio-Rickman-Väisälä; new approach to qr

Books
Lehto-Virtanen 1965, Ahlfors 1966: qc $n=2$
Caraman 1968, Väisälä 1971, qc, $n \ge 3$
Reshetnyak 1982, Vuorinen 1988, qr, $n \ge 3$

8.1. Def. A map $f: G \to \mathbb{R}^n$ is called quasiregular (qr) if f is ACL^n and $\exists\, K \ge 1$ s.t.

(8.2) $$|f'(x)|^n \le K\, J_f(x)\ ; \quad |f'(x)| = \sup\{|f'(x)h| : |h| \le 1\}$$

for a.e. $x \in G$. The least number in (8.2) is called the outer dilatation of f and denoted by $K_O(f)$. If f is qr, then $\exists\, K \ge 1$ s.t.

(8.3) $$J_f(x) \le K\, \ell(f'(x))^n\ ; \quad \ell(f'(x)) = \min\{|f'(x)h| : |h| = 1\}$$

holds a.e. $x \in G$ and the least number K in (8.3) is called the inner dilatation of f and denoted $K_I(f)$. The maximal dilatation is the number $K(f) = \max\{K_O(f), K_I(f)\}$. If $K(f) \le K$ we say that f is K-qr. If f is not qr we set $K_O(f) = K_I(f) = K(f) = \infty$.

Lin. algebra (cf. V7, p. 44, R12, p. 22) $\Longrightarrow$

(8.4) $$K_O(f) \le K_I(f)^{n-1}, \quad K_I(f) \le K_O(f)^{n-1}.$$

8.5. Lemma. Let $f: G \to \mathbb{R}^n$ be qr. Then

(1) f is either a constant or sp discr. open

(2) f is diffble a.e.

(3) f satisf. condition (N) i.e. if $A \subset G$ and $m(A)=0$, then $m(fA)=0$

(4) $m(B_f) = m(fB_f) = 0$

These results are due to Reshetnyak

8.6. Def. Let $G \subset \bar{\mathbb{R}}^n$ be a domain. A map $f: G \to \bar{\mathbb{R}}^n$ is termed quasimeromorphic (qm) if either $fG = \{\infty\}$ or the set $E = f^{-1}(\infty)$ is discrete and $f_1 = f | G \setminus (E \cup \{\infty\})$ is qr. We set $K(f) = K(f_1)$, $K_0(f) = K_0(f_1)$, $K_I(f) = K_I(f_1)$.

8.7. Def. If f is a homeo satisfying (8.2) and (8.3), with $J_f(x)$ replaced with $|J_f(x)|$, then f is called quasiconformal (qc).

8.8. Rmk. For $n=2$, $K=1$ the class of K-qr maps agrees with the class of anal. functions.

8.9. A second def. of a qc map. Let $G, G' \subset \bar{\mathbb{R}}^n$ be domains and $f: G \to G'$ a homeo. Then we say that f is K-qc if

(8.10) $\quad M(\Gamma)/K \le M(f\Gamma) \le K\, M(\Gamma)$

$\forall$ curve family Γ in G. Furthermore we define

$$K_I(f) = \sup \frac{M(f\Gamma)}{M(\Gamma)} \; ; \; K_0(f) = \sup \frac{M(\Gamma)}{M(f\Gamma)}$$

where the sup is taken over all such curve families Γ that $M(\Gamma)$ and $M(f\Gamma)$ are not simultaneously 0 or ∞. Thus

(8.11) $\quad M(\Gamma)/K_0(f) \le M(f\Gamma) \le K_I(f)\, M(\Gamma) \qquad \forall\, \Gamma$ in G.

The equivalence of the two defs 8.7 and 8.9 is proved in V7.

8.12. Ex. We show that (8.11) does not hold for anal. functions (hence not for qr maps)

(1) Let $f_k(z) = z^k$, $z \in \mathbb{C}$, $k = 1, 2, \ldots$ and $\Gamma = \Delta(S^1, S^1(1/e))$. Then

$M(\Gamma) = 2\pi$, $M(f_k \Gamma) \le 2\pi / \log(e^k) = 2\pi/k$.

Because f_k is anal, $K(f_k) = 1$. For $k \ge 2$ the left ineq. (8.11) fails.

(2) Let $f(z) = \exp(z)$, $z \in \mathbb{C}$, $A_t = \{(x,y) \in \mathbb{R}^2 : x = t\}$ and $\Gamma = \Delta(A_0, A_1)$. Then $f\Gamma \subset \Delta(S^1(e), S^1)$ so that

$$M(f\Gamma) \le 2\pi / \log e = 2\pi$$

On the other hand $M(\Gamma) = \infty$, cf. 3.37. $\therefore$ (8.11) fails for anal. f.

8.13. Thm Let $f: G \to \mathbb{R}^n$ be qr and $A \subset G$ Borel s.t. $N(f,A) < \infty$. If Γ is a curve family in A, then

$$M(\Gamma) \le N(f,A)\, K_0(f)\, M(f\Gamma)$$

Pf (Idea, see MRV1, §3). Let $L(x,f) = \limsup_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$

Let $\rho' \in F(f\Gamma)$, set $\rho : G \to \dot{\mathbb{R}}^1$, $\rho(x) = \rho'(f(x))\, L(x,f)$. Change of variables $\Rightarrow \rho \in F(\Gamma)$

$$M(\Gamma) \le \int \rho^n \, dm = \int_A \rho'(f(x)) L(x,f)^n \, dm \le K_0(f) \int_A \rho'(f(x))^n J_f(x)\, dm$$

$$= \int \rho'(y)^n N(y,f,A)\, dm(y) \le N(f,A) \int (\rho')^n \, dm$$

change of variables

8.14. Thm. Let $f : G \to \mathbb{R}^n$ be a non-const. qr map, $\Gamma \subset G$ a curve family in G, Γ' in $\mathbb{R}^n$ and $m \ge 1$ an integer s.t. the foll. holds: There exists a set $E_0 \subset G$, $m(E_0) = 0$ s.t. $\forall$ path β in Γ' has paths $\alpha_1, \ldots, \alpha_m \in \Gamma$ s.t. $f \circ \alpha_j \subset \beta$, $j = 1, \ldots, m$ and $\forall x \in G \setminus E_0$ $x \in |\alpha_j|$ at most for one index j. Then $M(\Gamma') \le K_I(f) M(\Gamma)/m$.

Theorems 8.13, 8.14 are called the K_0 and K_I inequal. 8.14 is due to Poleckii P1 and Väisälä V8.

Note that it is not required in 8.14 that $f\Gamma = \Gamma'$. In many applications we have $f\Gamma < \Gamma'$. If D is a normal domain, if Γ' is a path family in fD and if Γ is the family of all such curves α in D that $f \circ \alpha \in \Gamma'$, then the hypotheses of 8.14 are fullfilled with $m = N(f, D)$ and $E_0 = B_f$ (cf. 7.27 and 8.5 (4)).

The connection 5.5 between the modulus of a curve family and the capacity of a curve family enables one to formulate the K_0- and K_I-inequalities for capacities. Observe first that if $f: G \to \mathbb{R}^n$ is discr. open and (A, C) a condenser in G, then (fA, fC) is a condenser in fG. If A is a normal domain of f, then (A, C) is called a normal condenser.

8.15. <u>Thm.</u> Let $f: G \to \mathbb{R}^n$ be a non-const. qr map. Then

(1) $$\operatorname{cap}(fA, fC) \le K_I(f) \operatorname{cap}(A, C) / M(f, C)$$

for all condensers (A, C) in G where

$$M(f, C) = \inf_{y \in fC} \sum_{x \in C \cap f^{-1}(y)} i(x, f).$$

Furthermore, for all normal condensers (A, C) we have

(2) $$\operatorname{cap}(A, C) \le K_0(f)\, N(f, A) \operatorname{cap}(fA, fC).$$

8.16. <u>Thm</u>. Let $f: G \to \mathbb{R}^n$ be non-const. qr. Then

(1) $J_f(x) > 0$ a.e. $x \in G$

(2) If $g: G' \to \mathbb{R}^n$ is qr and $fG \subset G'$, then

$$K_0(g \circ f) \le K_0(f) K_0(g) \text{ and } K_I(g \circ f) \le K_I(f) K_I(g).$$

<u>The proofs</u> of 8.15 and 8.16 are in MRV1

8.17. Ex. (1) A linear bijection is qc V7, 16.1

(2) The radial map $x \mapsto |x|^{a-1}x$, $a \neq 0$, is qc and $K_I(f) = |a|$, $K_O(f) = |a|^{n-1}$ if $|a| \geq 1$, $K_I(f) = |a|^{1-n}$, $K_O(f) = 1/|a|$ if $|a| \leq 1$ V7, 16.2

(3) Let (r, φ, z) be the cyl. coord. in $\mathbb{R}^3$, $D_\alpha = \{(r, \varphi, z) \in \mathbb{R}^3 : 0 < \varphi < \alpha\}$. For $0 < \alpha < \beta \leq 2\pi$, the map $f: D_\alpha \to D_\beta$, $f(r, \varphi, z) = (r, \beta\varphi/\alpha, z)$ is called folding, $K(f) = (\beta/\alpha)^{n-1}$ V7, 16.3

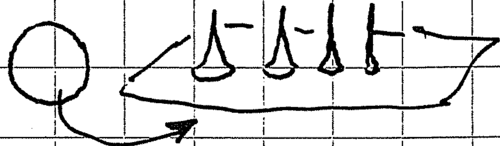
(4) Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be L-bilipschitz in eucl. metric. Then $K(f) \leq L^{2(n-1)}$ V7, 34.2.

8.18. Ex. A domain D that can be mapped onto B^n by a qc map $f: D \to B^n$ is called qclly equivalent to B^n.

(1) A bounded convex domain is qclly equivalent to B^n. The unbounded domain $\{x \in \mathbb{R}^3 : 0 < x_1 < 1\}$ is not [GV].

(3) The cylinder $B^{n-1} \times \mathbb{R}$ is qclly equiv. to B^n.

(4) If D is conformally equiv. to B^n, then ∂D is either a sphere or a hyperplane if $n \geq 3$ and hence smooth (fails for $n = 2$). In fact, this follows from Liouville's thm: 1-qc map $f: D \to D'$, $D, D' \subset \mathbb{R}^n$, $n \geq 3$, is of the form $f = g|D$, $g \in GM(\overline{\mathbb{R}}^n)$, see G2, R3, R6, BI1. For K-qc maps, $K > 1$ B^n can be mapped onto a non-smooth domain

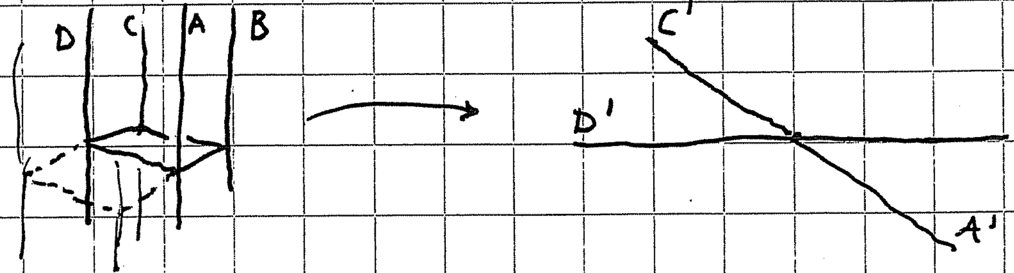


8.19. Ex. The winding map

$(r, \varphi, z) \mapsto (r, k\varphi, z)$, $k = 2, 3, \ldots$

is qr

8.20. <u>The Zorich map</u> First map the cylinder $[-1,1]\times[-1,1]\times\mathbb{R}$ qc:lly onto the upper half space H^3



Consider mapping an adjacent cylinder, symm. to the original, (w.r.t. one face) onto the lower half space. Continue this process by tiling $\mathbb{R}^2$ into squares of side 2, and map the cylinders ~~with~~ these squares as bases onto upper/lower half spaces so as to obtain a cont. map $f:\mathbb{R}^3\to\mathbb{R}^3$. The branch set is the union of the lines parallel to the z-axis and passing through the corners of each square

Squares are marked with + (−) if they are mapped into upper (lower) half space

+	−	+
−	+	−
+	−	+

The standard procedure for applying the K_O and K_I inequal. is to find a contradiction e.g. in the form $a < a < \infty$. For this purpose the basic question is how to find the curve family Γ? Unfortunately there is no general recipe for this.

There is however "a universal choice of the curve family" that applies to many cases. More precisely, the conf. invariants μ_G and λ_G are based on such choices.

8.21. <u>Thm</u>. If $f: G \to \mathbb{R}^n$ is a non-const. qr mapping, then

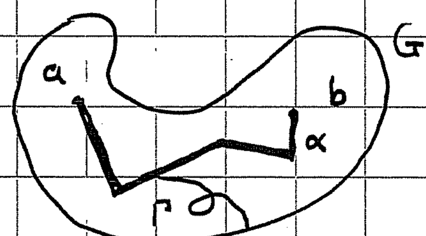
$$\mu_{fG}(f(a), f(b)) \le K_I(f)\, \mu_G(a,b) \quad \forall\, a,b \in G. \tag{1}$$

In particular, $f : (G, \mu_G) \to (fG, \mu_{fG})$ is Lipschitz-cont. If $N(f,G) < \infty$, then for $\forall a, b \in G$, $f(a) \neq f(b)$,

$$\lambda_G(a,b) \le K_0(f)\, N(f,G)\, \lambda_{fG}(f(a), f(b)). \tag{2}$$

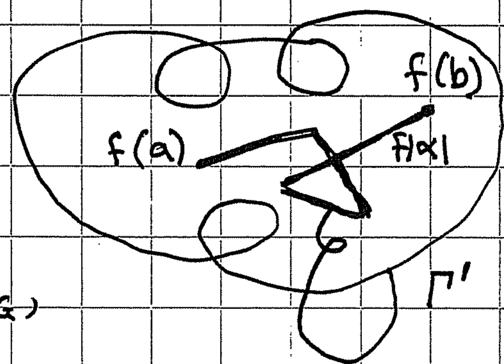
<u>Proof.</u> (1) Fix $a, b \in G$ and a path $\alpha : [0,1] \to G$ s.t. $\alpha(0) = a$, $\alpha(1) = b$ and let $\Gamma' = \{\beta : [0,1] \to \mathbb{R}^n \mid \beta(0) \in |f \circ \alpha|, \beta(1) \in \partial fG\}$. Let Γ be the family of all such maximal liftings, which start at a point of $|\alpha|$. I.e. $\gamma \in \Gamma \Leftrightarrow \exists\, \beta \in \Gamma'$ s.t. γ is the max. lifting of β and $\gamma(0) \in |\alpha|$. Then $f\Gamma < \Gamma'$ and we get by 8.14

$$\mu_{fG}(f(a), f(b)) \le M(\Gamma') \le M(f\Gamma) \le K_I(f)\, M(\Gamma)$$



Because $\overline{|\beta|} \cap \partial fG \neq \emptyset$ $\forall \beta \in \Gamma'$, by 7.25 we see $\overline{|\gamma|} \cap \partial G \neq \emptyset$ $\forall \gamma \in \Gamma$. I.e.

$$M(\Gamma) \le M(\Delta(|\alpha|, \partial G; G)).$$



This inequality with the previous one yields, in view of the def. of μ_G, the desired conclusion.

(2) Let $\beta_j : [0,1) \to fG$ be paths s.t. $\beta_1(0) = f(a)$, $\beta_2(0) = f(b)$, $\beta_j(t) \to \partial fG$, $t \to 1$, $j = 1,2$ and $|\beta_1| \cap |\beta_2| \cap fG = \emptyset$. Let $\gamma_j : [0, c_j) \to G$ be the max. lifting of β_j s.t. $\gamma_1(0) = a$, $\gamma_2(0) = b$. Because $\beta_j(t) \to \partial fG$ when $t \to 1$, it follows from 7.25 that $\gamma_j(t) \to \partial G$ when $t \to c_j$, $j = 1,2$. Let $\Gamma = \Delta(|\gamma_1|, |\gamma_2|; G)$. Now $8.13 \Rightarrow$

$$\lambda_G(a,b) \le M(\Gamma) \le K_0(f)\, N(f,G)\, M(f\Gamma)$$

Because β_1, β_2 were arb. paths as specified above and because $f\Gamma \subset \Delta(|\beta_1|, |\beta_2|; fG)$, the proof follows from

8.22. Cor. If $f: G \to fG$ is qc, then for all $a, b \in G$

(1) $\quad \mu_G(a,b)/K_0(f) \le \mu_{fG}(f(a), f(b)) \le K_I(f)\, \mu_G(a,b)$

(2) $\quad \lambda_G(a,b)/K_0(f) \le \lambda_{fG}(f(a), f(b)) \le K_I(f)\, \lambda_G(a,b), \quad a \neq b$

8.21 says that a qr $f: G \to \mathbb{R}^n$ is a Lipschitz between the (pseudo) metric spaces $f: (G, \mu_G) \to (fG, \mu_{fG})$. It would be desiderable to replace (G, μ_G) with a less abstract space such as (G, k_G) or (G, j_G). This will be our goal.

8.23. Thm Let $E \subset \overline{\mathbb{R}}^n$ be compact and $f: B^n \to \overline{\mathbb{R}}^n \setminus E$ a non-const. K-qm mapping. If $c(E) > 0$ then for $x, y \in B^n$, $x \neq y$,

$$q(f(x), f(y)) \le \frac{aK}{c(E)}\, \mu_{B^n}(x,y) \le \frac{bK}{c(E)} \left(-\log \operatorname{th} \frac{\rho(x,y)}{2}\right)^{1-n}$$

where a and b only depend on n.

Pf. 3.48 (4), (5) $\Rightarrow$

$$\mu_{fB^n}(f(x), f(y)) \ge d_4 \min\{c(E), d_3\, q(f(x), f(y))\} \ge d_4\, q(f(x), f(y)) \min\{d_3, c(E)\}$$

Because $c(E) \le d_2 < \infty$ by 3.48 (3) we have $d_3 \ge d_3\, c(E)/d_2$ and

$$\mu_{fB^n}(f(x), f(y)) \ge d_4\, c(E)\, q(f(x), f(y)) \min\{1, d_3/d_2\}.$$

By 6.7 (1) and (5.24)

$$\mu_{B^n}(x,y) = \gamma_n\left(1/\operatorname{th}\left(\frac{\rho(x,y)}{2}\right)\right) \le \omega_{n-1} \left(\log \frac{1}{\operatorname{th}(\rho(x,y)/2)}\right)^{1-n}$$

The pf follows from these inequalities and 8.22 (1) with the constants

$$a = 1/(d_4 \min\{d_3, c(E)\}) \text{ and } b = \omega_{n-1}\, a.$$