III QUASIREGULAR MAPS

Goals: 1) To define qc and qr maps $f: G \to fG$

2) To study these as unif. cont. maps between metric spaces $f: (G, \mu_G) \to (fG, \mu_{fG})$ [also wrt. to other metrics]

3) To show that the results in 2) imply the qr counterpart of the Schwarz lemma, along with many other results.

7. Discrete and open mappings

Sources: MRV1, V4, V5, RR, s. 110–190, RI1

In this section all maps will be continuous.

7.1. Def. The set T^n, $n = 2, 3, \ldots$ consists of ordered triples (y, f, D), where $f: G \to \overline{\mathbb{R}}^n$ is cont., $G \subset \overline{\mathbb{R}}^n$ is a domain, D is a domain s.t. $\overline{D} \subset G$ and $y \in \overline{\mathbb{R}}^n \setminus f(\partial D)$.

7.2. Lemma (RR) There exists a unique function $\mu: T^n \to \mathbb{Z}$, the topol. degree s.t.

(1) $y \mapsto \mu(y, f, D)$ is a constant in each component of $\overline{\mathbb{R}}^n \setminus f(\partial D)$

(2) $|\mu(y, f, D)| = 1$, if $y \in fD$ and $f|\overline{D}$ is an injection

(3) $\mu(y, id, D) = 1$, if $y \in D$ and id is the identity map

(4) Let $(y, f, D) \in T^n$ and $D_1, \ldots, D_k$ disjoint domains s.t. $(y, f, D_j) \in T^n$ and $f^{-1}(y) \cap D \subset \bigcup_{j=1}^{k} D_j$. Then $\mu(y, f, D) = \sum_{j=1}^{k} \mu(y, f, D_j)$.

(5) Let $(y, f, D), (y, g, D) \in T^n$ s.t. $\exists$ a homotopy, a cont. map $h_t: \overline{D} \to \overline{\mathbb{R}}^n$, $t \in [0, 1]$ with $h_0 = f|\overline{D}$, $h_1 = g|\overline{D}$ and $(y, h_t, D) \in T^n$ $\forall\, t \in [0, 1]$. Then $\mu(y, f, D) = \mu(y, g, D)$.

7.3. Lemma. (1) If $(y, f, D) \in T^n$ and $y \notin f\overline{D}$, then $\mu(y, f, D) = 0$.
(2) If f is a constant c, then $\mu(y, f, D) = 0 \quad \forall y \neq c$.
(3) If f is diffble at $x_0 \in D$ and $J_f(x_0) = \det f'(x_0) \neq 0$, then there exists a nbd U of x_0 s.t. $(y, f, U) \in T^n$ and $\mu(y, f, U) = \operatorname{sign} J_f(x_0)$ for $y \in fU$.

Pf. (1) Choose domains $D_1, D_2 \subset D$ with $D_1 \cap D_2 = \emptyset$. By 7.2(4) $\mu(y, f, D_1) = \mu(y, f, D) = \mu(y, f, D_2)$. On the other hand 7.2(4) $\Rightarrow \mu(y, f, D) = \mu(y, f, D_1) + \mu(y, f, D_2) = 2\mu(y, f, D) \Rightarrow \square$
(2) Follows from (1)
(3) See RR

If $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$, then $\mu(y, f, \overline{\mathbb{R}}^n)$ is defined for every $y \in \overline{\mathbb{R}}^n$ and its value does not depend on y (7.2(1)). It is denoted simply by $\mu(f)$.

7.4. Lemma. If p is a reflection in $\{x : x_n = 0\}$ then $\mu(p) = -1$.

Pf. Let $q(x) = (x_1, \ldots, x_{n-1}, |x_n|)$, $q(\infty) = \infty$. Let $D_1 = B^n(e_n, \frac{1}{2})$ $D_2 = B^n(-e_n, \frac{1}{2})$. Because $q^{-1}(e_n) = \{e_n, -e_n\}$ we have by 7.3 (1)

$$0 = \mu(q) = \mu(e_n, q, \overline{\mathbb{R}}^n) \underset{7.2(4)}{=} \mu(e_n, q, D_1) + \mu(e_n, q, D_2)$$

$$= \mu(e_n, id, D_1) + \mu(e_n, p, D_2) = 1 + \mu(p).$$

7.5. Lemma. Let $(y, f, D), (y, g, D) \in T^n$ s.t. $f|\partial D = g|\partial D$ and $\infty \notin f\overline{D} \cup g\overline{D}$. Then $\mu(y, f, D) = \mu(y, g, D)$.

Pf. Set $h_t(x) = t f(x) + (1-t) g(x)$ in 7.2(5).

7.6. <u>Rmk.</u> The assumption $\infty \notin f\bar{D} \cup g\bar{D}$ is essential.
Counterex.: $D = R^n_+$, $f = id$, g = reflection in ∂D, when
$\mu(e_n, id, D) = 1 \neq 0 = \mu(e_n, g, D)$.

We next generalize the definition of a sense-preserving map.

7.7. <u>Def.</u> A map $f: G \to \bar{R}^n$ is termed <u>sense-preserving</u> (sense-reversing), abbr. sp (sr) if $\mu(y, f, D) > 0$ (<0) $\forall$ domain $D \subset G$ with $\bar{D} \subset G$ and $\forall y \in fD \setminus f(\partial D)$.

7.8. <u>Lemma</u> (RR, V4) Let $f: G \to \bar{R}^n$ and $g: fG \to \bar{R}^n$ be maps and set $h = g \circ f$. If f and g both are sp or sr, then h is sp. If one is sp and the other one is sr then h is sr.

7.9. <u>Lemma</u> Let $(y, f, D) \in T^n$ and $f_j: \bar{D} \to \bar{R}^n$ a sequence of maps such that $f_j \to f$ unif. (in the q-metric). Then $\mu(y, f_j, D) \to \mu(y, f, D)$.

<u>Pf.</u> Fix j_0 s.t. $q(f_j(x), f(x)) < q(y, f(\partial D))$ $\forall j \geq j_0$ $\forall x \in \bar{D}$.
Then $y \notin f_j(\partial D)$ i.e. $(y, f_j, D) \in T^n$ $\forall j \geq j_0$. Now we can define a homotopy, transforming f to f_j, $j \geq j_0$.
7.2(5) $\Longrightarrow$ claim.

7.10. <u>Ex.</u> Can you define the homotopy in this proof by a formula?

7.11. <u>Rmk.</u> In RR the degree theory is built on the foundation of algebraic topology. An alternative approach is based on approximation by C^∞ - functions, for which the degree $\mu(y, f, D)$

can be defined as the sum (cf. DE, RE12)

$$\mu(y,f,D) = \sum_{z\in f^{-1}(y)\cap D} \operatorname{sign} J_f(x) .$$

Recall that a bijection $f: X \to Y$ is called a <u>homeomorphism</u> if both f and f^{-1} are continuous. A cont. map $f: X \to Z$ is said to be a <u>local</u> <u>homeom</u>. at $x \in X$ if there are neighborhoods $U \ni x$ and $V \ni f(x)$ such that $f|U : U \to V$ is a homeom. The <u>branch</u> <u>set</u> of a map $f: G \to \overline{\mathbb{R}}^n$ is $B_f = \{z \in G : f \text{ is not a local homeo at } z\}$. It is clear that B_f is closed in G. We say that f is <u>open</u>, if fA is open in $\overline{\mathbb{R}}^n$ whenever $A \subset G$ is open, <u>light</u> if $f^{-1}(y)$ is totally disconnected (i.e. consists of point components only) $\forall y \in fG$ and <u>discrete</u> if $f^{-1}(y)$ is isolated $\forall y \in fG$.

V5

The next result due to Černavskii is crucial, see also Väisälä

7.12. Thm (CE1, CE2) Let $f: G \to \overline{\mathbb{R}}^n$ be discrete open. Then $\dim B_f = \dim fB_f = \dim f^{-1}(fB_f) \le n-2$ (dim = the topol. dimension).

7.13. <u>Rmk.</u> It is clear that $G \setminus B_f$ is open. Because $\dim B_f \le n-2$ it follows, see HW, that $G\setminus B_f$ is connected whenever G is. For $n=2$ one can prove more: B_f is isolated (Stoilow). For $n \ge 3$ one can prove that B_f does not contain a single isolated point.

Let $G \subset \overline{\mathbb{R}}^n$ be a domain. Denote $J(G) = \{D : D$ is a domain such that $\overline{D} \subset G\}$.

7.14 <u>Def.</u> Let $f: G \to \overline{\mathbb{R}}^n$ be discr, $x \in G$, and $U \in J(G)$ s.t. $x \in U$ and $\{x\} = (f^{-1}(f(x))) \cap \overline{U}$. The number

$$i(x,f) = \mu(f(x), f, U)$$

is called the <u>local index</u> of f at x.

7.15. <u>Rmk.</u> Clearly $(f(x), f, U) \in T^n$ above. Next we show that this def. is independent of U. Let U_1, U_2 be two nbds of x with the above properties. Then $7.2 \Rightarrow$

$$\mu(f(x), f, U_j) = \mu(f(x), f, Q(x,r)) + \overbrace{\mu(f(x), f, U_j \setminus Q(x,r))}^{=0}$$
$$= \mu(f(x), f, Q(x,r)) = \mu(f(x), f, Q(x,r'))$$

$\forall r' \in (0, q(x, \partial U_j))$ $\forall r \in (0, r']$ $\Rightarrow$ independence

Let $f: G \to \mathbb{R}^n$ be discr. open. Then 7.12, 7.13 $\Rightarrow G \setminus B_f$ is connected and $i(x,f)$ is a constant in $G \setminus B_f$, either $+1$ or -1. In the first case f is sp, in the second case sr. In both cases we see by 7.2(4) that for $D \in J(G)$ and $y \in fD \setminus f(\partial D)$ $D \cap f^{-1}(y) = \{x_1, \ldots, x_k\}$

(7.16) $$\mu(y, f, D) = \sum_{j=1}^{k} i(x_j, f) \; .$$

A domain $D \in J(G)$ is termed a <u>normal domain</u> if $\partial fD = f(\partial D)$. A <u>normal neighborhood</u> of x is a normal domain D with $\{x\} = \overline{D} \cap f^{-1}(f(x))$

7.17. <u>Rmk.</u> If $f: G \to \mathbb{R}^n$ be open and $D \in J(G)$. Then $\partial fD \subset f(\partial D)$. This was one of the hw problems.

In function theory the local index is called the <u>winding number</u> (see BU s. 84, WH2).

7.18. Lemma. Let $f: G \to \mathbb{R}^n$ be open, $U \subset \mathbb{R}^n$ domain and D a component of $f^{-1}U$ s.t. $D \in J(G)$. Then D is a normal domain, $fD = U$ and $fD \in J(fG)$.

Let $f: G \to \mathbb{R}^n$, $x \in G$, $r > 0$. The x-component of $f^{-1}B^n(f(x), r)$ is denoted by $U(x, f, r)$.

7.19. Lemma. Let $f: G \to \mathbb{R}^n$ be discr. open. Then $\lim_{r \to 0} d(U(x,f,r)) = 0$ $\forall x \in G$. If $U(x,f,r) \in J(G)$ then $U(x,f,r)$ is a normal domain and $fU(x,f,r) = B^n(f(x), r) \in J(fG)$. Furthermore, for every $x \in G$ $\exists\, \sigma_x > 0$ s.t. for $\forall r \in (0, \sigma_x]$

(1) $U(x,f,r)$ is a normal neighb. of x

(2) $U(x,f,r) = U(x,f,\sigma_x) \cap f^{-1}B^n(f(x), r)$.

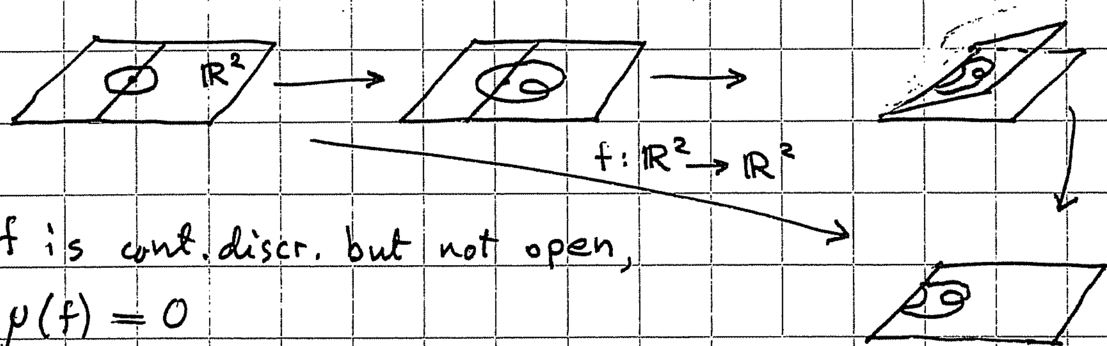
(3) $\partial U(x,f,r) = U(x,f,\sigma_x) \cap f^{-1}S^{n-1}(f(x), r)$ if $r < \sigma_x$

(4) $\overline{\mathbb{R}}^n \setminus U(x,f,r)$ is connected

(5) $\overline{\mathbb{R}}^n \setminus \overline{U}(x,f,r)$ —"—

(6) If $0 < r < s \le \sigma_x$ then $\overline{U}(x,f,r) \subset U(x,f,s)$ and $U(x,f,s) \setminus \overline{U}(x,f,r)$ is a ring.

7.20. Example (cf. 7.17) A case when $\partial fD \subset f(\partial D)$ fails.



f is cont. discr. but not open,

$\mu(f) = 0$

For $f: G \to \overline{\mathbb{R}}^n$, $A \subset \mathbb{R}^n$, $y \in \overline{\mathbb{R}}^n$ write

$$N(y,f,A) = \operatorname{card}\{A \cap f^{-1}(y)\};$$

$$N(f,A) = \sup\{N(y,f,A) : y \in \overline{\mathbb{R}}^n\} \;;\; N(f) = N(f,G).$$

The number $N(y,f,A)$ is called the <u>multiplicity</u> of y in A and the number the <u>maximal</u> <u>multiplicity</u> in A.

Recall that the notation $f: G \to \mathbb{R}^n$ usually includes the assumptions (a) $G \subset \mathbb{R}^n$ is a domain (b) f is cont.

7.21. <u>Lemma.</u> Let $f: G \to \mathbb{R}^n$ be sp, discr. and open.

(1) If $D \in J(G)$, then $N(y,f,D) \le \mu(y,f,D)$ $\forall y \in \mathbb{R}^n \setminus f\partial D$ and $N(y,f,D) = \mu(y,f,D)$ $\forall y \in \mathbb{R}^n \setminus fA$, $A = \partial D \cup (D \cap B_f)$.

(2) If D is a normal domain, then $N(f,D) = \mu(f,D)$.

(3) If $A \subset G$ is compact, then $N(f,D) < \infty$.

(4) $\forall x \in G$ $\exists$ nbd V s.e. if $U \subset V$ and U is a nbd of x, then $N(f,U) = i(x,f)$.

(5) $x \in B_f$ iff $i(x,f) \ge 2$.

7.22. <u>Rmk.</u> (1) 7.21(4) $\Rightarrow i(x,f) = \lim_{r \to 0} N(f, B^n(x,r))$.

A basic example is the function $f: B^2 \to B^2$, $f(z) = z^2$ with $i(0,f) = 2$.

(2) Let $A_j \subset \mathbb{R}^n$ and $f: G \to \mathbb{R}^n$ cont. One can show that

$$N(y,f,\cup A_j) \le \sum N(y,f,A_j),$$

$$N(f,\cup A_j) \le \sum N(f,A_j).$$

If A is a Borel set in G, then $N(y,f,A)$ is measurable, see RR, pp 216-219.

7.23. <u>Path lifting.</u> Let $f: G \to \overline{\mathbb{R}}^n$ and let $\beta: [a,b) \to \overline{\mathbb{R}}^n$ be a path and $x_0 \in G$ s.t. $f(x_0) = \beta(a)$. We say that $\alpha: [a,c) \to G$ is a <u>maximal lifting</u> of β starting at x_0 if

$$(7.24)\begin{cases} (1) & \alpha(a) = x_0 \\ (2) & f \circ \alpha = \beta \mid [a,c) \\ (3) & \text{If } c < c' \le b, \text{ then } \nexists\, \alpha': [a,c') \to G \text{ s.t.} \\ & \alpha = \alpha' \mid [a,c) \text{ and } f \circ \alpha' = \beta \mid [a,c'). \end{cases}$$

If $\beta: [a,b) \to \overline{\mathbb{R}}^n$ is a path and if $C \subset \overline{\mathbb{R}}^n$, then we write $\beta(t) \to C$ when $t \to b$ if $q(\beta(t), C) \to 0$ when $t \to b$.

7.25. <u>Lemma</u> Let $f: G \to \overline{\mathbb{R}}^n$ be light and open, $x_0 \in G$, $\beta: [a,b) \to \overline{\mathbb{R}}^n$ path s.t. $\beta(a) = f(x_0)$ and either $\exists \lim_{t \to b} \beta(t)$ or $\beta(t) \to \partial fG$ when $t \to b$. Then β has a maximal lifting starting at x_0, $\alpha: [a,c) \to G$. If $\alpha(t) \to x_1 \in G$ when $t \to c$ then $c = b$ and $f(x_1) = \lim_{t \to b} \beta(t)$. Otherwise $\alpha(t) \to \partial G$ when $t \to c$. If f is discr. open and $i(\alpha(t), f)$ is a const. when $t \in [a,c)$, then α is the only lifting of β starting at x_0.

7.18, 7.19, 7.21 are from MRV1, 6.23, 6.25 from MRV3.

7.26. <u>Rmk.</u> Let $f: G \to \mathbb{R}^n$ be light and open, $x_0 \in G$ and $\beta: [0,1] \to \overline{\mathbb{R}}^n$ be a path s.t. $\beta(0) = f(x_0)$ and $\beta(1) \in \partial fG$. Then 7.25 $\Rightarrow \beta$ has a max. lifting starting at x_0, $\alpha: [0,c) \to G$ with $\alpha(t) \to \partial G$, when $t \to c$.

We say that $f: G \to \mathbb{R}^n$ is <u>proper</u> if $f^{-1}K$ is a compact subset of G whenever $K \subset fG$ is compact and closed if fC is closed in fG whenever $C \subset G$ is closed in G. If f is discr. open and D a normal domain then $f|D$ is closed.

7.27. Lemma Let $f: G \to \mathbb{R}^n$ be discr. open. FAE

(1) f is proper

(2) f is closed

(3) $N(f,G) = p < \infty$ and $\forall y \in fG$, $p = \sum_{j=1}^{k} i(x_j; f)$, $\{x_1, \ldots, x_k\} = f^{-1}(y)$.

As the example $z \mapsto z^2$ shows, a path lifting starting at a branch point need not be unique. For 7.27 see V5, MSR1, VU1.

7.28. Lemma Let $f: G \to \mathbb{R}^n$ be discr. open, closed, $p = N(f,G) < \infty$ and let $\beta: [a,b) \to fG$ be a path. Then $\exists\, \alpha_j: [a,b) \to G$, $1 \le j \le p$, s.t.

(1) $f \circ \alpha_j = \beta$

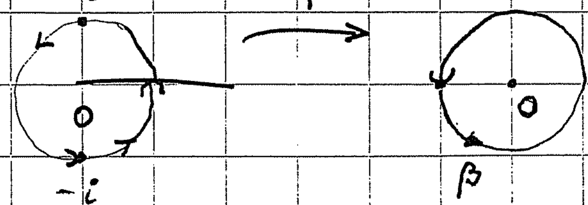
(2) $\operatorname{card}\{j : \alpha_j(t) = x\} = |i(x,f)|\ \forall x \in f^{-1}|\beta|\ \forall t \in [a,b)$

(3) $\bigcup_{j=1}^{p} |\alpha_j| = f^{-1}|\beta|$

7.29. Ex. Let $U = \mathbb{R}^2 \setminus \{s e_1 : s \ge 0\}$, $f: z \mapsto z^2$ and $\beta(t) = e^{i\pi t}$, $1 \le t \le 3$. Now $fU = \mathbb{R}^2 \setminus \{0\}$, $|\beta| \subset fU$. Now β has maximal liftings $\alpha_1 : [1,3] \to U$, $\alpha_2 : [1, 4/3) \to U$ starting at i and -1 resp.

$\alpha_1(t) = e^{i\frac{\pi}{2}t}$, $1 \le t \le 3$

$\alpha_2(t) = e^{i\frac{3\pi}{2}t}$, $1 \le t \le \frac{4}{3}$



7.30. The winding map. Let (r,φ,z), $r\geq 0$, $\varphi\in[0,2\pi)$, $z\in\mathbb{R}$, be the cylindrical coordinates in $\mathbb{R}^3$. Fix an integer $p>1$ and consider the map $f(r,\varphi,z)=(r,p\varphi,z)$. Then f is discrete open and $N(f,B^3)=p$. In fact, the z-axis $\{(r,\varphi,z)\in\mathbb{R}^3 : r=0\}$ is B_f and $fB_f=B_f$ and $N(y,f,B^3)=p$ for every $y\in B^3\setminus(fB_f)$. In some respects, f resembles the power $z\mapsto z^p$.

7.31. Simply connected domain. Recall that a domain $D\subset\mathbb{R}^n$ is simply connected iff every path $\alpha:[0,1]\to D$ with $\alpha(0)=\alpha(1)$ can be continuously deformed to the point $\alpha(0)$ in D.

Clearly $B^2\setminus\{0\}$ is not simply connected. Observe that $B^n\setminus\{0\}$ is simply connected if $n\geq 3$.

7.32. Topological fact (monodromy theorem, covering map) Let $f:G\to fG$, $G, fG\subset\mathbb{R}^n$, be a closed local homeomorphism and fG be simply connected. Then f is 1-to-1, i.e. injective.

7.33. Open problem. Let $f:B^n\to fB^n$ be discrete, open, closed, $n\geq 3$, and let B_f be compact. Is it true that f is 1-to-1? (I first found this problem myself, working on my PhD thesis ca 1975 and it is one of the open problems on p. 193 of CGQM.)