<u>Proof.</u> Let C_x, C_y be as in (6.3). Then $d(C_x, C_y) \le |x-y|$ and $d(C_x) \ge d(x)$, $d(C_y) \ge d(y)$. The proof follows from 5.41.

6.29. <u>Rmk.</u> As in 5.41, 5.42 we see that

$$\tau(4s^2+4s) \ge \max\{2^{1-n}\tau(s),\ c_n \log(1+1/s)\}.$$

6.30. <u>Lemma.</u> Let $G \subset \mathbb{R}^n$ be a domain s.t. ∂G is connected. Then for $x, y \in G$, $x \neq y$

$$\mu_G(x,y) \ge \tau(4m^2+4m) \ge c_n\, j_G'(x,y)$$

where $c_n > 0$ is as in 3.33 and $m = \min\{d(x), d(y)\}/|x-y|$.

<u>Pf.</u> Fix $C = C_x$ as in (6.4). Then $d(C_{xy}) < d(\partial G)$ and $d(C_{xy}, \partial G) \le d \min\{d(x), d(y)\}$ yielding

$$d(C_{xy}, \partial G)/\min\{d(C_{xy}), d(\partial G)\} \le \min\{d(x), d(y)\}/|x-y|$$

and the first ineq. follows from 5.41. The 2nd ineq. $\Leftarrow$ 5.43.

6.31. <u>Rmk.</u> For c-QED domains 6.28 gives quite a good estimate. E.g. if $G = B^n$ then G is a $\frac{1}{2}$-QED and 6.28 $\Rightarrow$

$$\lambda_{B^n}(x,0) \ge \tfrac{1}{2}\tau\Big(\big(1+\tfrac{2|x|}{1-|x|}\big)^2 - 1\Big) \overset{5.20}{=} 2^{-n}\gamma\Big(\frac{1+|x|}{1-|x|}\Big)$$

$$6.7(2) \Rightarrow \lambda_{B^n}(x,0) = \frac{1}{2}\tau\Big(\frac{|x|^2}{1-|x|^2}\Big) = 2^{-n}\gamma\Big(\frac{1}{\sqrt{1-|x|^2}}\Big) \quad \Big(\ge 2^{-n}\gamma\Big(\frac{1+|x|}{1-|x|}\Big)\Big)$$

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6.32. <u>The next step.</u> The above results, such as 6.28 and 6.30, are adequate for several purposes as we shall see. An esthetic flaw e.g. in 6.30 is that m is only similarity invariant. This motivates the search for a Möb. inv. version of 6.30.

For a domain $G \subsetneq \overline{\mathbb{R}}^n$, $\operatorname{card}(\overline{\mathbb{R}}^n \setminus G) \geq 2$ and $b, c \in G$ let

$$m_G(b,c) = \sup\{|a,b,d,c| : a,d \in \partial G\}.$$

It is easy to see that

(a) $m_G(b,c) = m_G(c,b)$

(b) $m_G(b,c) \to 0 \iff b \to c$

(c) b fixed, $c \to \partial G \Rightarrow m_G(b,c) \to \infty$

(d) $m_G(b,c) = m_{fG}(f(b), f(c)) \ \forall f \in GM \ \forall a,b \in G$

(e) $G_1 \subset G_2 \Rightarrow m_{G_1}(b,c) \geq m_{G_2}(b,c) \ \forall b,c \in G_1$

(f) $m_G(b,c) \geq q(\partial G)\, q(b,c)$

6.33. <u>Lemma</u> $m_{B^n}(b,c) = \exp(\rho_{B^n}(b,c)) - 1$

<u>Pf.</u> $GM(B^n)$-inv $\Rightarrow$ may ass. $b = -re_1 = -c$. Then $\rho(b,c) = 2\log\frac{1+r}{1-r}$ by (2.13) and hence $r = \operatorname{th}\frac{\rho(b,c)}{4}$. For $a, d \in \partial B^n$

$$|a,b,d,c| \leq \frac{2|b-c|}{(1-r)^2} = \frac{4\operatorname{th}\frac{\rho}{4}}{(1-\operatorname{th}\frac{\rho}{4})^2} \quad \text{with} = \text{if } a = -e_1 = -c.$$

Therefore

$$m_{B^n}(b,c) = 4\operatorname{th}\frac{\rho}{4}\left(\frac{\operatorname{ch}\frac{\rho}{4}}{\operatorname{sh}\frac{\rho}{4} - \operatorname{ch}\frac{\rho}{4}}\right)^2 = \frac{4\operatorname{sh}\frac{\rho}{4}\operatorname{ch}\frac{\rho}{4}}{(e^{-\rho/4})^2} = 4e^{\frac{\rho}{2}}\operatorname{sh}\frac{\rho}{2} = e^{\rho(b,c)} - 1$$

Define $\quad \rho_G(b,c) = \log(1 + m_G(b,c)).$

Let $G = \mathbb{R}^n \setminus \{0\}$. Now for $x, y \in G$

$$\left.\begin{array}{l} |0,x,\infty,y| = |x-y|/|x| \\ |\infty,x,0,y| = |x-y|/|y| \end{array}\right\} \; m_G(x,y) = \frac{|x-y|}{\min\{|x|,|y|\}}$$

$\Rightarrow j_G(x,y) = \rho_G(x,y)$ for $G = \mathbb{R}^n \setminus \{0\}$.

The next result is due to Seittenranta Math. Proc. Camb. Phil. Soc 1999

6.34. <u>Thm.</u> ρ_G is a metric.

A natural next step would be to re-formulate the results in this and preceding sections replacing j_G with ρ_G. To be done...

6.35. Rmk. One can show that the inequality $\mu_G \geq c_n j_G$ in 6.30 cannot be reversed: $\nexists$ const. $c < \infty$ such that $\mu_G \leq c\, j_G$. Next we show, however, that there is a majorant for μ_G in terms of k_G.

6.36. Lemma. Let G be a domain in $\mathbb{R}^n$, $s \in (0,1)$, $x, y \in G$. If $k_G(x,y) \leq 2\log(1+s)$, then

(1) $$\mu_G(x,y) \leq \gamma\left(1/\operatorname{th}\left(\frac{k_G(x,y)}{1-s}\right)\right).$$

Moreover, there exist $b_1 = b_1(n)$ and $b_2 = b_2(n)$ s.t.

(2) $$\mu_G(x,y) \leq b_1\, k_G(x,y) + b_2, \quad \forall\, x, y \in G.$$

Pf. (1) Choose a quasihyp. geod. segment $J_G[x,y]$ connecting x to y and let $z \in J_G[x,y]$ with $k_G(x,y) = 2k_G(x,z) = 2k_G(y,z)$. By 2.30,

$$j_G(x,z) \leq k_G(x,z) \leq \log(1+s)$$

and hence $x \in B^n(z, s\,d(z))$ by (2.33). Let $B_z = B^n(z, d(z))$. By 2.32

$$k_{B_z}(x,z) \leq \log\left(1 + \frac{|x-z|}{d(z) - |x-z|}\right) \leq \log\left(1 + \frac{|x-z|}{(1-s)d(z)}\right)$$
$$\leq \frac{1}{1-s}\log\left(1 + \frac{|x-z|}{d(z)}\right) \leq \frac{1}{1-s} j_G(x,z) \leq \frac{1}{1-s} k_G(x,z).$$

By the choice of z we have a similar bound for $k_{B_z}(z,y)$. Hence

$$k_{B_z}(x,y) \leq k_{B_z}(x,z) + k_{B_z}(z,y)$$
$$\leq \frac{1}{1-s}\left(k_G(x,z) + k_G(z,y)\right) = \frac{1}{1-s} k_G(x,y).$$

Denote by ρ_{B_z} the hyperbolic metric of B_z. Then

$$\rho_{B_z}(x,y) \leq 2k_{B_z}(x,y) \leq \frac{2}{1-s} k_G(x,y)$$

Therefore ($\rho = \rho_{B_z}$)

$$\mu_G(x,y) \leq \mu_{B_z}(x,y) = 2^{n-1}\tau\left(\frac{1}{\operatorname{sh}^2\frac{\rho(x,y)}{2}}\right) = \gamma\left(1/\operatorname{th}\frac{\rho(x,y)}{2}\right)$$
$$\leq \gamma\left(1/\operatorname{th}\left(\frac{k_G(x,y)}{1-s}\right)\right).$$

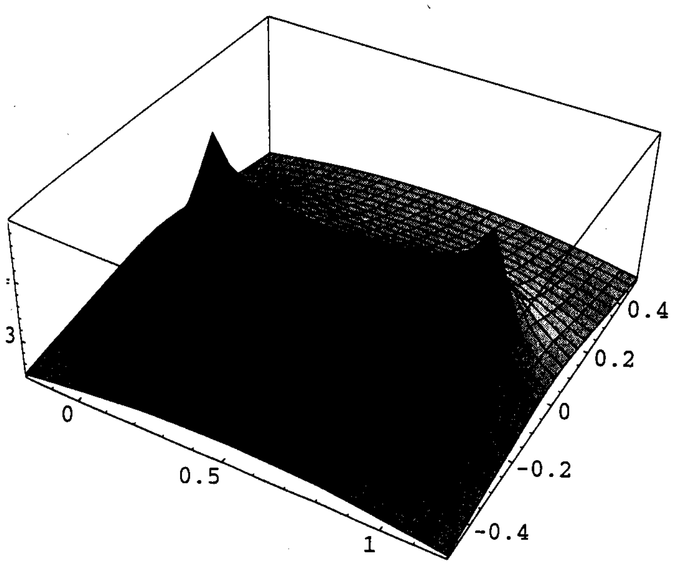
(2) By Lemma 4.9/CGQM there are points $x_1,\dots,x_{p+1}\in J_G[x,y]$ with $x_1=x$, $x_{p+1}=y$ and $k_G(x_j,x_{j+1}) = 2\log(1+s)$ for $j=1,\dots,p-1$ and $k_G(x_p,x_{p+1}) < 2\log(1+s)$ and $p\le 1+k_G(x,y)/(2\log(1+s))$. Then by part (1)

$$\mu_G(x,y)\le\sum_{j=1}^{p} \mu_G(x_j,x_{j+1})\le p\, b_2$$

where $b_2 = \mu\left(1/\operatorname{th}\left[[2\log(1+s)]/(1-s)\right]\right)$. We may choose $b_1 = b_2/(2\log(1+s))$.

6.37. Rmk. For $n=2$ Thm 6.17 can be further refined. In fact, $p(z)$ may be computed numerically. The book comes with a diskette containing programs for the computation of functions such as $\mathcal{K}(r)$, $\mu(r)$, $\mu^{-1}(y)$, $p(z)$ in the languages C, MATLAB, and Mathematica. Some graphics produced by Mathematica programs appear below.

Surface $z=p(x+iy)$



Contour lines of $p(x+iy)$