QUASICONFORMAL AND QUASIREGULAR MAPPINGS ①

by M. Vuorinen, vuorinen@utu.fi, Fall 2007

This graduate course provides a gateway to some topics of contemporary analysis, crucial to Finnish math.
Prerequisites: Basics of measure theory, metric spaces
Two ways of completing the course: a) exam
b) seminar lecture. In both cases, homework gives you extra bonus.

Lectures:

Demonstrations:

Main sources:

M. Vuorinen: Conformal Geometry and Qr Mappings, 1988 [CGQM]
J. Väisälä: Lectures on n-Dim. Qc Mappings, 1971
+ material on the course www-page

Other books:

Anderson-Vamanamurthy-Vuorinen: 1997 AVVB
Lehto-Virtanen: 1973

Vocabulary, abr: qc = quasiconformal
qr = quasiregular

In Finnish: qc → kk, qr → ks

Notation: $B^n(x,r) = \{z \in \mathbb{R}^n : |x-z| < r\}$; $B^n(r) = B^n(0,r)$
$S^{n-1}(x,r) = \partial B^n(x,r)$; $S^{n-1}(r) = S^{n-1}(0,r)$
$B^n = B^n(1)$, $S^{n-1} = \partial B^n$

We indentify $\mathbb{R}^2$ with $\mathbb{C}$ $(x,y) \to x+iy$; $i = (0,1)$

0. Review and introduction

Some basic facts will be reviewed from metric spaces, complex variables etc.

Maps: $f : \underbrace{A}_{\text{domain}} \longrightarrow \underbrace{B}_{\text{image domain, target}}$ also $f : z \longmapsto z^2 + 3z$, $z \in \mathbb{C}$

Ex. $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. $f(z) = z^{1/4}$, $g(z) = (z-1)^{1/2}$. Find the set $g(f(H^2))$.

Metric space X a space, $m : X \times X \to [0, \infty)$ function

a) $m(x,y) = m(y,x) \geq 0 \quad \forall x,y \in X$

b) $m(x,y) \leq m(x,z) + m(z,y) \quad \forall x,y,z \in X$

c) $m(x,y) = 0 \iff x = y$

If a) and b) hold, then m is a pseudometric. If also c) holds then m is a metric and (X, m) a metric space. Balls are denoted: $B_m(x,r) = \{z \in X : m(z,x) < r\}$

Ex. (1) $\mathbb{R}_+ = (0,\infty)$, $\sigma(x,y) = \left|\log \frac{x}{y}\right|$, $x,y \in \mathbb{R}_+$ is a metric

(2) $G \subset \mathbb{R}^n$ open, $d(x) = d(x, \partial G)$, $h(x,y) = \left|\log \frac{d(x)}{d(y)}\right|$

Open sets If (X, m) is a m.s. and $A \subset X$, then A is open if $\forall x \in A\ \exists r > 0 : B_m(x,r) \subset X$. E.g. $B_m(x_0, 1)$ is an open set. A set $F \subset X$ is closed if $F = X \setminus A$ for some open set A. A domain is an open connected set (for connectedness, see metric spaces course). Open subset denoted $A \Subset X$.

Topology Let τ be a collection of subsets of X. τ is a topology if

(1) $A_i \in \tau$, $i \in I \Rightarrow \bigcup A_i \in \tau$

(2) $A_1, \dots, A_m \in \tau \Rightarrow \bigcap_{i=1}^m A_i \in \tau$

(3) $\emptyset, X \in \tau$

If τ_m is the collection of all open subsets of X, then τ_m is a topology (defined by the metric m). This is the standard topology we use for X.

<u>Open map</u>. $f: (X, m_1) \to (Y, m_2)$ is <u>open</u> if fA is open whenever $A \subset X$ is open and <u>closed</u> if fK is closed whenever $K \subset X$ is closed. In general these are not equivalent (but for bijective maps they are).

A bijective map $f: X \to Y$ is a homeomorphism if f and f^{-1} are continuous. f is discrete if for every $y \in Y$, $f^{-1}(y)$ consists of isolated points.

<u>Invariance of domain</u> (deep result): $U \Subset \mathbb{R}^n$, $f: U \to \mathbb{R}^n$ continuous and injective $\Rightarrow fU \Subset \mathbb{R}^n$.

Jordan domain

<u>Cross-ratio</u>: $z_j, j = 1, \dots, 4 \in \mathbb{C}$: $(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_2} \frac{z_2 - z_4}{z_3 - z_4}$.

<u>Möbius transformations</u> $z \mapsto \frac{az+b}{cz+d}$; $ad - bc \neq 0$

Special cases: rotation : $z \mapsto e^{i\theta} z$

translation : $z \mapsto z + a$

; $z \mapsto 1/z$

Given the points z_1, z_2, z_3 and w_1, w_2, w_3 the M transf. w with $w(z_k) = w_k$) is given by

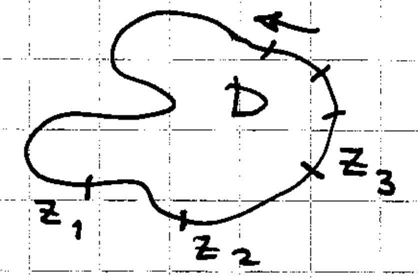
$$\frac{z_1 - z_3}{z_1 - z_2} \frac{z_2 - z}{z_3 - z} = \frac{w_1 - w_3}{w_1 - w_2} \frac{w_2 - w}{w_3 - w}$$

The mapping $z \mapsto \bar{z}$ is called a reflection.

Möbius transformations preserve cross-ratio !

Möbius transformations are conformal.

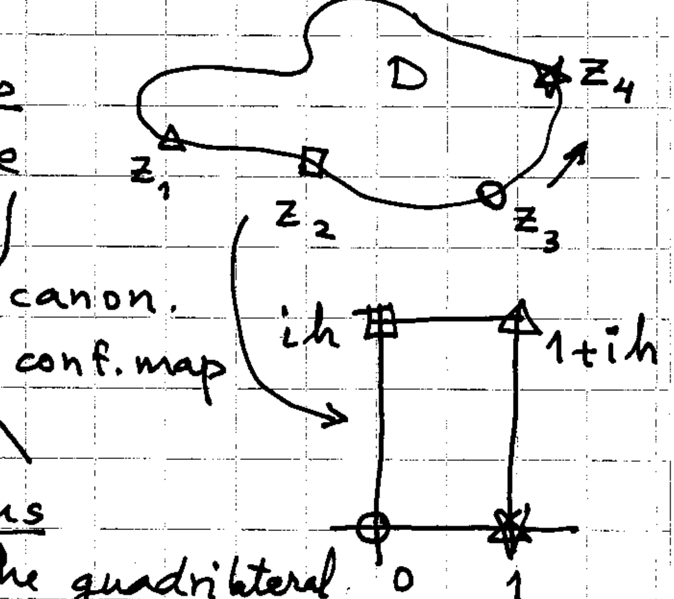
<u>Oriented Jordan domains</u> Let $D \subset \mathbb{C}$ be a Jordan domain. Then ∂D is homeom to the circle S^1. Let z_k, $k=1,\dots,p$ be points of ∂D. We say that $(D; z_1,\dots,z_p)$ is a positively oriented domain, if moving along ∂D the points $z_1,\dots,z_p$ occur in this order and if the domain D is on the left hand side.



<u>Conformal maps preserve orientation.</u>
Let $D_1, D_2 \subset \mathbb{C}$ be Jordan domains, $h: \overline{D}_1 \to \overline{D}_2$ a homeomorphism s.t. $h|D_1 : D_1 \to D_2$ is conformal. If $(D_1; z_1,\dots,z_p)$ is pos. oriented so is $(D_2; h(z_1),\dots,h(z_p))$.

<u>A quadrilateral</u> is a Jordan domain D with four boundary points z_1, z_2, z_3, z_4 such that $(D; z_1, z_2, z_3, z_4)$ is positively oriented.

The <u>canonical conformal map</u> maps D onto a rectangle with vertices z_1, z_2, z_3, z_4 corresponding to $1+ih, ih, 0, 1$ where h is uniquely determined.



It is called the <u>modulus</u> $M(D; z_1, z_2, z_3, z_4)$ of the quadrilateral.

A simplified definition: $M(D; z_1, z_2, z_3, z_4) = \tau(s)/2$ if $(D; z_1, z_2, z_3, z_4) \overset{conf}{\approx} (H^2; -1, 0, s, \infty)$, $s>0$.
Here τ is a transcendental function (given later).
Naturally $h = \tau(s)/2$.