Introduction to Conformal Geometry and Quasiconformal Maps
Department of Mathematics and Statistics
University of Helsinki
Winter 2011 / Vuorinen
Exercise 11, 2011-04-12 `File: icg1111.tex, 2011-4-4,9.28`

1. Let $c \geq 1$, and let $G \subset \mathbf{R}^n$ be an open set. A positive, continuous function $u : G \to R_+ \setminus \{0\}$ is called a *c-Harnack function* if the inequality

$$\sup_{\mathbf{B}(x,r)} u(z) \leq c \inf_{\mathbf{B}(x,r)} u(z)$$

holds whenever $\mathbf{B}(x, 2r) \subset G$. Well known examples of functions satisfying Harnack's inequality are positive harmonic functions in the plane.
(a) Let $u(z) = \arg z$ and $G = \mathbf{C} \setminus \{x \in \mathbf{R} \colon x \geq 0\}$. Find a constant $c \geq 1$ such that $u(z)$ is c-Harnack in G.
(b) Let $K \subset G$ be compact and $u(z)$ as in (a). Does there exist a constant D depending on $d(K)/d(K, \partial G)$ such that

$$u(z_1) \leq D\, u(z_2)$$

for all $z_1, z_2 \in K$? Hint. Show that in the domain of part (a) there are compact sets K such that the quasihyperbolic diameter of K does not have a majorant in terms of $d(K)/d(K, \partial G)$.
(c) Let $K \subset G$ be compact. Show that $u(x) = \exp(-k_G(x, K))$ satisfies the Harnack inequality.

2. Let $G, G' \subset \mathbf{R}^n$ and $f : (G, k_G) \to (G', k'_G)$ be uniformly continuous, and let $b' \in \partial G'$. Show that $u : G \to R_+, u(x) = |f(x) - b'|$ satisfies Harnack's inequality.

3. Let $E \subset \mathbf{B}$ be compact. Suppose that

$$m_n(E_k) = a_k,\ E_k = \{x \in \mathbf{R}^n \setminus E : 2^{-k-1} < d(x, E) < 2^{-k}\}, k = 1, 2, \ldots .$$

Use Lemma 5.24[CGQM] to find an upper bound for $\mathsf{M}(\Delta(E, S^{n-1}(2)))$. Apply your bound to give a sufficient condition for $\operatorname{cap} E = 0$ in terms of the numbers (a_k).

4. Let $G = \mathbf{B} \setminus \{0\}, f : G \to G' = f(G)$, be a homeomorphism with the property that there exist curves $\alpha_j : [0, 1) \to G, j = 1, 2,$ such that $\alpha_j(t) \to 0, f(\alpha_j(t)) \to \beta_j \in \partial G', t \to 1$. Show that $\beta_1 = \beta_2$ if there exists $C \geq 1$ with $k_{G'}(f(x), f(y)) \leq C k_G(x, y)$ for all $x, y \in G$. Show that $\beta_1 = \beta_2$ also holds if there exists $K \geq 1$ such that $\mathsf{M}(\Gamma) \leq K\mathsf{M}(f\Gamma) \leq K^2\mathsf{M}(\Gamma)$ for all curve families Γ in G.

5. Let $D = \{z \in \mathbf{C} \colon 0 < \arg z < \theta, 0 < |z| < 1\}$, $z_1 = 1$, $z_2 = e^{i\alpha}$ for $0 < \alpha < \theta$, $z_3 = e^{i\theta}$ and $z_4 = 0$. Find the modulus of the quadrilateral $(D; z_1, z_2, z_3, z_4)$.
In other words, find a conformal map of D onto $\{z \in \mathbf{C} \colon \mathrm{Im} z > 0\}$ such that the points z_k are mapped onto the real axis and compute the cross ratio of these points.