Introduction to Conformal Geometry and Quasiconformal Maps
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1. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous. Then f is said to be *open* if it maps all open subsets onto open subsets of G', *closed* if it maps all closed subsets onto closed subsets of G', and *proper* if for every compact $K \subset G'$ also $f^{-1}K$ is compact. Note the condition $fG = G'$ above, i.e. f is a surjective map.
(a) Show that the map $f\colon H \to B^2 \setminus \{0\}$, $H = \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$, $f(z) = \exp(z)$, is open but neither proper nor closed.
(b) Prove: Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. If $y \in G'$, then $f^{-1}(y)$ is compact.
(c) Prove: Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. If $y \in G'$ and U is an open neighborhood of $f^{-1}(y)$ in G, then there is an open neighborhood V of y in G' such that $f^{-1}V \subset U$.

2. Let $G, G' \subset \overline{\mathbf{R}}^n$ be domains, and let $f\colon G \to G' = fG$ be continuous, open, and closed. Then f is proper, i.e., for every compact $E \subset G'$, also $f^{-1}E$ is compact.

3. For $\alpha > 0$ we denote by $I(\alpha)$ the class of compact subsets E of $\overline{\mathbf{B}}^n$ with

$$A = \int_{B^n(2)\setminus E} \frac{dm}{d(x,E)^\alpha} < \infty.$$

Then, for example, $\{0\} \in I(\alpha)$ when $\alpha < n$, and $S^{n-1} \in I(\alpha)$ when $\alpha < 1$. Fix $E \in I(\alpha)$, denote $E_k = \{x \in \mathbf{R}^n : 2^{-k-1} \le d(x,E) \le 2^{-k}\}$, $k = 1, 2, \ldots$, and for $p > 0$ let Γ_p be the family of all curves in $\Delta(E, S^{n-1}(2); \mathbf{R}^n)$ with $\ell(\gamma \cap E_k) \ge 2^{-kp}$. Show that $\mathsf{M}(\Gamma_p) = 0$ for $p \in (0, \alpha/n)$.

4. Let $x, y \in \mathbf{B}^n$, $x \neq y$ and $M \in (0, \frac{\rho(x,y)}{2})$. Show that

$$\mathsf{M}(\Delta(D(x,M), D(y,M); \mathbf{B}^n)) \ge d_1(n,M)\rho(x,y)^{1-n},$$

where $d_1 > 0$.
5. Let $f : \mathbf{B}^n \to \mathbf{B}^n$ be a homeomorphism mapping each sphere centered at 0 onto another sphere centered at 0 (such a mapping is called a *radial mapping*) and with the property that for some $K \ge 1, \mathsf{M}(\Gamma)/K \le \mathsf{M}(f(\Gamma)) \le K\mathsf{M}(\Gamma)$ whenever Γ is the family of all curves connecting the boundary components of a spherical annulus centered at 0. Show that for all $x \in \mathbf{B}^n$

$$|x|^{1/\alpha} \le |f(x)| \le |x|^\alpha, \alpha = K^{1/(1-n)}.$$

6. Let $G = \mathbb{B}^2 \setminus \{0\}$.
(a) For $0 < r < 1/2$ compute the quasihyperbolic area w.r.t. k_G of the annulus $\{z : r < |z| < 1/2\}$.
(b) For $1/2 < r < 1$ compute the quasihyperbolic area w.r.t. k_G of the annulus $\{z : 1/2 < |z| < r\}$.