Introduction to Conformal Geometry and Quasiconformal Maps
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Winter 2011 / Vuorinen
Exercise 7, 2011-03-15 `File: icg1107.tex, 2011-3-14,8.16`

1. Let $t > r > s > 0$, $E \subset \mathbf{B}^n(s)$ and $\Delta_a = \Delta(E, S^{n-1}(a))$. Show that $\mathsf{M}(\Delta_r) \le c\mathsf{M}(\Delta_t)$, where c is only dependent on n, r, s and t.

2. Prove Theorem 6.1 (3) [CGQM].

3. Let $G = \mathbf{R}^n \setminus \{0\}$ and let s_G be defined as

$$s_G(x,y) = \frac{|x-y|^2}{2|x||y|} \;, \quad x,\, y \in G \;.$$

Define ρ_G by $\operatorname{ch} \rho_G(x,y) = 1 + s_G(x,y)$. Show that

$$j_G(x,y)/2 \le \rho_G(x,y) \le 4\, j_G(x,y)$$

for $x, y \in G$. Hint: Use the inequality from h0405 to the effect that for $a \ge 0$

$$\begin{aligned} \log\left(1+\max\left\{a, \sqrt{a}\,\right\}\right) &\le b \le \log\left(1+a+\sqrt{a}\,\right) \\ &\le 2\log\left(1+\max\left\{a, \sqrt{a}\,\right\}\right) \end{aligned}$$

if $\operatorname{ch} b = 1 + \frac{1}{2}a$.

4. Show that for $0 < r < 1$ and $M > 0$, $m_h\big(\bigcup_{|x|\le r} D(x,M)\big) \le d_2(n,M)(1-r)^{1-n}$, where m_h is the hyperbolic measure of $(\mathbf{B}, \rho_{\mathbf{B}})$.

5. Show that for given $\varepsilon > 0$ there are numbers $r_1 > s_1 > r_2 > s_2 > ...$ such that $\mathsf{M}(\Delta(E,F,\mathbf{R}^n)) < \varepsilon$, when $E = \cup S^{n-1}(r_j)$ and $F = \cup S^{n-1}(s_j)$.

6. Let $G, G' \subset \mathbf{R}^n$, $n \ge 2$, and let $f : G \to G'$ be a homeomorphism with the following property: There exists $c_1 \in (0,\infty)$ such that for every subdomain $D \subset G$ and for all $x, y \in D$,

$$(\star) \qquad j_{fD}(f(x), f(y)) \le c_1 j_D(x,y).$$

Show that for each $z \in G$,

$$H(f,z) = \limsup_{r\to 0} \left\{ \frac{|f(x)-f(z)|}{|f(y)-f(z)|} \;:\; |x-z| = |y-z| = r \right\} \le c_2,$$

where $c_2 \in (1,\infty)$. Show that this inequality holds (possibly with a different constant c_2) also if in $(\star)$ j_D and j_{fD} are replaced by k_D and k_{fD}.