Introduction to Conformal Geometry and Quasiconformal Maps
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1. (1) Show that an inversion f in $S^{n-1}(a,r)$, when $a_n = 0$, preserves the upper half-space
$$f(\mathbf{H}^n) = \mathbf{H}^n.$$
(2) Show that the expression
$$\frac{|x-y|^2}{2x_n y_n},$$
where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, is invariant under an inversion in $S^{n-1}(a,r)$ when $a_n = 0$.

2. Let f be the translation $f : x \mapsto x + b$. Find an upper bound for the Lipschitz constant of f in spherical metric. (Hint: Lemma 1.54 (3)[CGQM].)

3. Verify the following elementary relations.
(1) $1 - e^{-s} \le \operatorname{th} s \le 1 - e^{-2s}$ for $s \ge 0$.
(2) If $s \ge 0$, then
$$\operatorname{th} s = \frac{\operatorname{th} 2s}{1 + \sqrt{1 - \operatorname{th}^2 2s}} \;.$$
Further, if $u \in [0,1]$ and $2s = \operatorname{arth} u$, then
$$\operatorname{th} s = \frac{u}{1 + \sqrt{1-u^2}} \le \tfrac{1}{2}(u + u^3) \;.$$
(3) $\log \operatorname{th} s = -2 \operatorname{arth} e^{-2s}$, $s > 0$.
(4) $\log(1+x) \le \operatorname{arsh} x \le 2\log(1+x)$, $x \ge 0$
(5) $2\log\big(1 + \sqrt{\tfrac{1}{2}(x-1)}\,\big) \le \operatorname{arch} x \le 2\log\big(1 + \sqrt{2(x-1)}\,\big)$, $x \ge 1$.

4. Observe first that, for $t \in (0,1)$,
$$\rho_{\mathbf{H}^n}(te_n, e_n) = \rho_{\mathbf{H}^n}\big(te_n,\, S^{n-1}(\tfrac{1}{2}e_n, \tfrac{1}{2})\big).$$
Making use of this observation and the formula for ρ-balls in terms of euclidean balls show that
$$B^n(\tfrac{1}{2}e_n, \tfrac{1}{2}) = \bigcup_{t \in (0,1)} D(te_n,\, \log\frac{1}{t}) \;.$$

5. Assume that $a \geq 0$ and define b by $\operatorname{ch} b = 1 + \frac{1}{2}a$. Show that

$$\begin{aligned} \log\left(1+\max\left\{a, \sqrt{a}\,\right\}\right) &\leq b \leq \log\left(1+a+\sqrt{a}\,\right) \\ &\leq 2\log\left(1+\max\left\{a, \sqrt{a}\,\right\}\right) \,. \end{aligned}$$

6. (1) Show that for distinct points a, b, c, u, v in $\mathbb{R}^n$,

$$|u,a,b,v| = |u,a,c,v||u,c,b,v|,$$
$$|u,a,b,v||u,b,a,v| = 1 = |u,a,b,v||v,a,b,u|.$$

(2) Conclude from (1) that, for a proper subdomain domain G of $\mathbb{R}^n$ and for $x, y \in G$, the quantity

$$m_G(x,y) \equiv \log\sup\{|u,x,y,v| : u,v \in \partial G\}$$

is nonnegative and symmetric, and that it satisfies the triangle inequality

$$m_G(x,y) \leq m_G(x,z) + m_G(z,y).$$

Observe also that $m_G(x,y) = m_{h(G)}(h(x),h(y))$ for $h \in \mathcal{GM}(G)$ and $x, y \in G$.
(3) Show that, for $x \in \mathbf{B}^n \setminus \{0\}, e_x = x/|x|$,

$$m_{\mathbf{B}^n}(0,x) = \log|-e_x, 0, x, e_x| = \log\left(\frac{1+|x|}{1-|x|}\right).$$

Conclude that $m_{\mathbf{B}^n}(x,y) = \rho_{\mathbf{B}^n}(x,y)$ for all x, y of points in $\mathbf{B}^n$.
(4) Show that m_G is not a metric for $G = \mathbb{R}^n \setminus \{0\}$.