

Introduction to Conformal Geometry and Quasiconformal Maps
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N.B. Numbered results/ formulas refer to CGQM.

1. Show that for all $a, x, y \in B^n$

$$\frac{|T_a x - T_a y|^2}{(1 - |T_a x|^2)(1 - |T_a y|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

Hint. Lectures, Ahlfors bracket.

2. Let $r \in (0, 1)$. Find a point $a \in (0, re_1)$ such that $T_a(0) = -T_a(re_1)$.

3. For $\varphi \in (0, \frac{1}{2}\pi)$, let $x_\varphi = (\cos \varphi, \sin \varphi)$ and $y_\varphi = (\cos \varphi, -\sin \varphi)$. Then there exists a Möbius transformation $T_a: \mathbf{B}^2 \rightarrow \mathbf{B}^2$ with $T_a e_1 = e_1$, $T_a(-e_1) = -e_1$, $T_a(x_\varphi) = e_2 = -T_a(y_\varphi)$. Find $|a|$.

4. Let $x, y \in \mathbb{R}^n$ and let t_x be a spherical isometry with $t_x(x) = 0$. Show that

$$|t_x y| = \frac{|x - y|}{\sqrt{(1 + |x|^2)(1 + |y|^2) - |x - y|^2}}.$$

Let $\alpha \in [0, \frac{1}{2}\pi]$ be such that $\sin \alpha = q(x, y)$. Then α is the angle between the segments $[e_{n+1}, t_x x] = [e_{n+1}, 0]$ and $[e_{n+1}, t_x y]$ at e_{n+1} . Show that the above formula can be rewritten as $|t_x y| = \tan \alpha$.

5. For $x, y \in \mathbf{B}^n$ and $T_x \in \mathcal{M}(\mathbf{B}^n)$ show that

$$|T_x y| = \frac{|x - y|}{\sqrt{|x - y|^2 + (1 - |x|^2)(1 - |y|^2)}} = \frac{s}{\sqrt{1 + s^2}},$$

where $s^2 = |x - y|^2 / ((1 - |x|^2)(1 - |y|^2))$.

6. Let $h: [0, \infty) \rightarrow [0, \infty)$ be strictly increasing with $h(0) = 0$ such that $h(t)/t$ is decreasing. Show that $h(x + y) \leq h(x) + h(y)$ for all $x > 0$. With $h(t) = t^\alpha$, $\alpha \in (0, 1)$, apply this result to show that if $d(x, y)$ is a metric then $d^\alpha(x, y) = d(x, y)^\alpha$ also is a metric.