Introduction to Conformal Geometry and Quasiconformal Maps
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Winter 2011 / Vuorinen
Exercise 1, 2011-01-24 `File: icg1101.tex, 2011-1-15,14.25`
N.B. Numbered results/ formulas refer to CGQM.

1. Perhaps the simplest non-injective map is $f\colon z \mapsto z^2$. Find a domain $D \subset \mathbb{C}$ such that $f|_D\colon D \to fD$ is not closed.

2. Find a Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

which maps $H^2 = \{(x,y) \in \mathbf{R}^2\colon y > 0\}$ onto $B^2 = \{(x,y) \in \mathbf{R}^2\colon x^2 + y^2 < 1\}$ such that $(-1,0,1) \mapsto (1,i,-1)$ $[i = (0,1)]$.

3. A mapping $f\colon \mathbf{R} \to \mathbf{R}$ is said to be Hölder continuous, if there exist constants $C, \beta > 0$ such that for all $x, y \in \mathbf{R}$ $|f(x) - f(y)| \le C|x-y|^\beta$. Let $f\colon \mathbf{R} \to \mathbf{R}$ be Hölder continuous with exponent $\beta > 1$. Show that f is a constant, equal to $f(0)$.

4. Let $\rho\colon \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $\mathbf{R}_+ = (0,\infty)$, be defined by $\rho(x,y) = |\log(x/y)|$. Show that ρ is a metric.
(a) Suppose that $u\colon (\mathbf{R}_+, \rho) \to (\mathbf{R}_+, \rho)$ is uniformly continuous. Show that $u(x) \le Ax^B$ for some constants A and B for all $x \ge 1$.
(b) Suppose that $u\colon (\mathbf{R}_+, d) \to (\mathbf{R}_+, \rho)$ is uniformly continuous, where d is the euclidean metric. Find an inequality of the same type as in (a) [but some other function in place of Ax^B].

5. Let $D = H^2 = \{(x,y) \in \mathbf{R}^2\colon y > 0\}$. The modulus $M(D; -1, 0, s, \infty)$ of a quadrilateral $(D; -1, 0, s, \infty)$, $s > 0$, is usually denoted $\tau(s)/2$ [we will define the function $\tau(s)$ later]. Fix $\alpha \in (0, \pi)$ and denote $D_\alpha = \{z \in \mathbf{C}\colon 0 < \arg z < \alpha\}$, $r > s > 0$, $t > u > 0$. Using the above notation, find $M(D_\alpha; re^{i\alpha}, se^{i\alpha}, u, t)$. [Hint: The modulus is a conformal invariant. Apply an auxiliary Möbius transformation to map the example to the previous case.]

6. Let $G, G' \subset \mathbb{R}^n$ be domains and $f : G \to G'$ be a homeomorphism. For $x, y \in G$ define $m_f(x,y) = |f(x) - f(y)|$. Is it true that m_f is a metric?