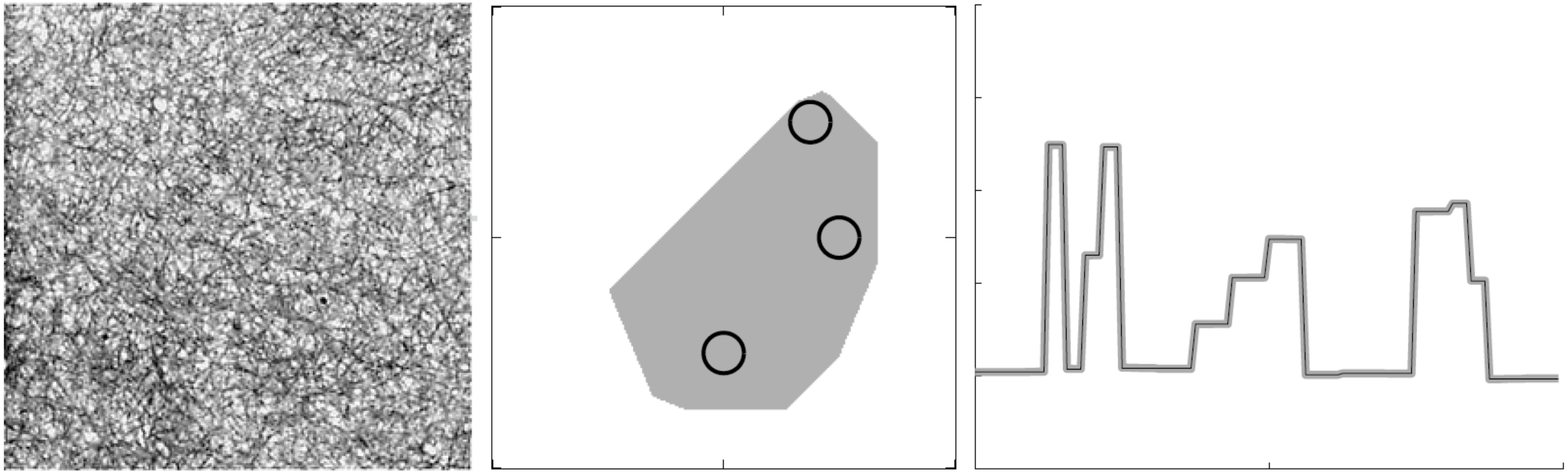
# Inverse Problems



Spring 2011
Department of Mathematics and Statistics
University of Helsinki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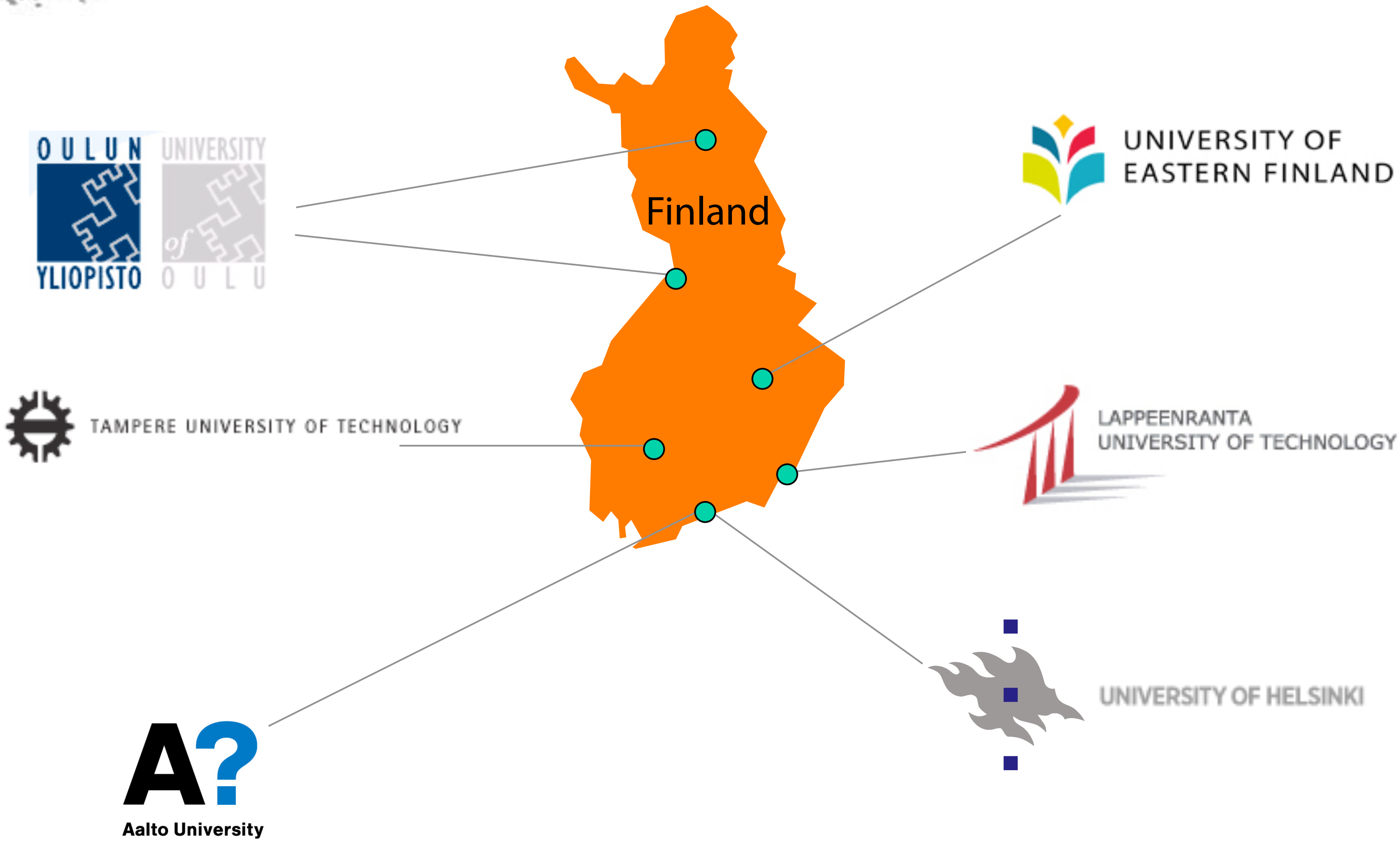
**Samuli Siltanen** (computational inversion)
Assistants: **Esa Niemi** and **Lauri Oksanen**

# Finnish Centre of Excellence in Inverse Problems Research




http://wiki.helsinki.fi/display/inverse/Home

# What are inverse problems?

# Inverse problem: Image deblurring

# Direct and inverse problem of image deblurring

**Direct problem:**

**Given a sharp photograph, what would the blurred version of the image look like?**

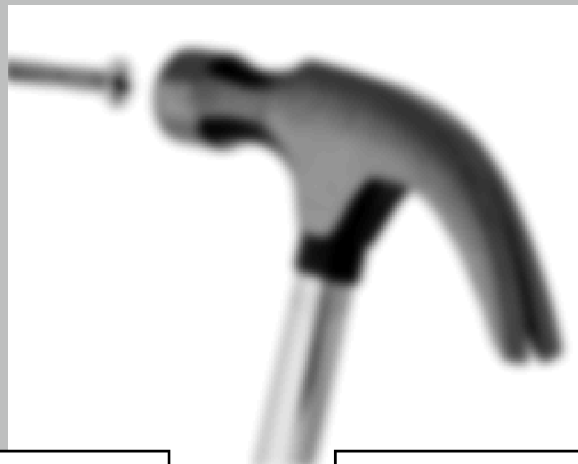
**Inverse problem:**

**Given a blurred photograph, reconstruct the sharp image**

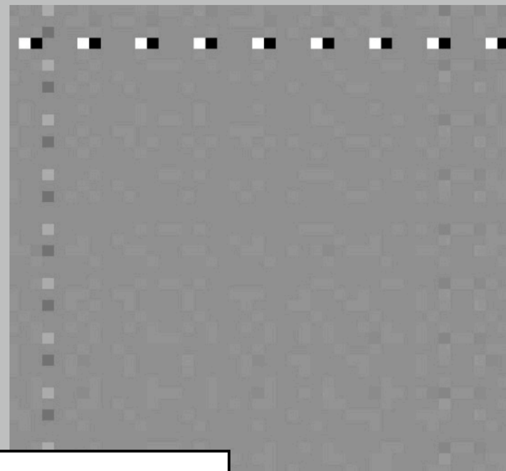
# The inverse problem is more difficult

Original image

Blurred image



Simple inversion

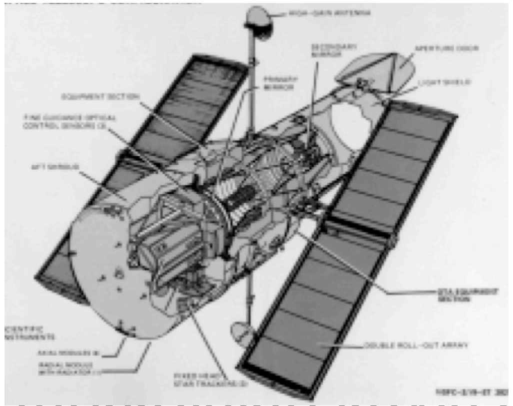


Direct problem

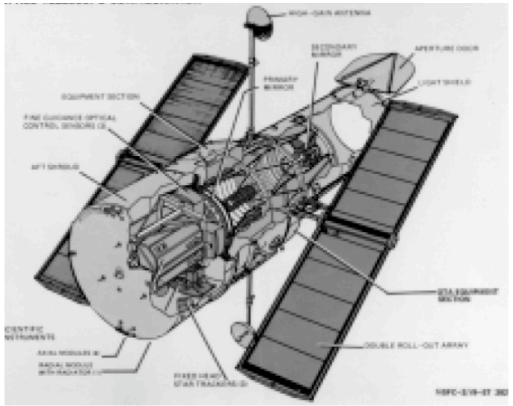
Inverse problem

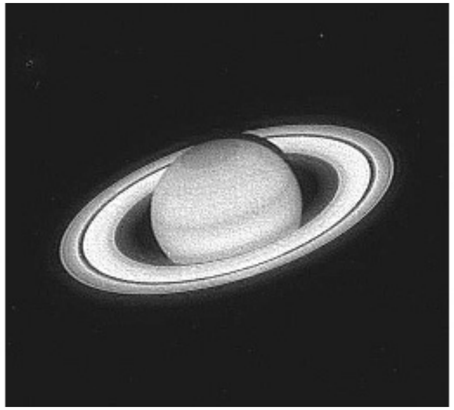
## With properly regularized inversion we can sharpen the photograph

# The Hubble space telescope had a flaw in its mirror, resulting in blurred images



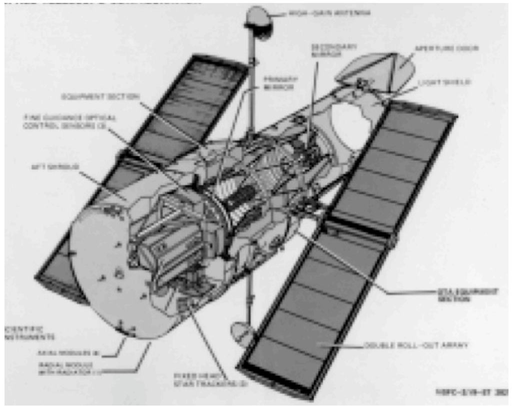

# The mirror flaw was compensated by a deconvolution algorithm

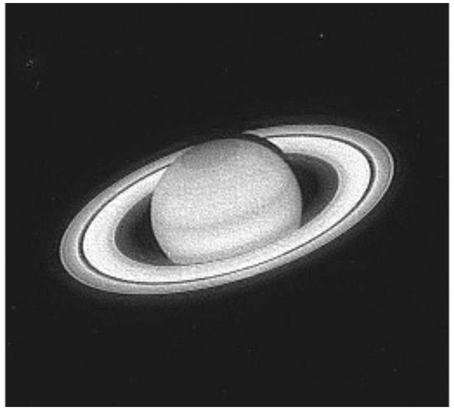


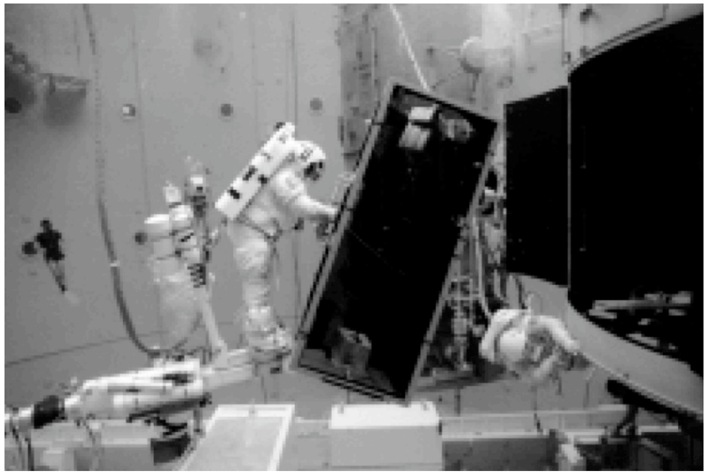
Source: NASA,  Quarktet

# The mirror flaw was compensated by a deconvolution algorithm





The mirror was replaced in 1993. However, even the new sharp images could be further enhanced with deconvolution!



Source: NASA,  Quarktet

# Inverse problem: Computerized tomography

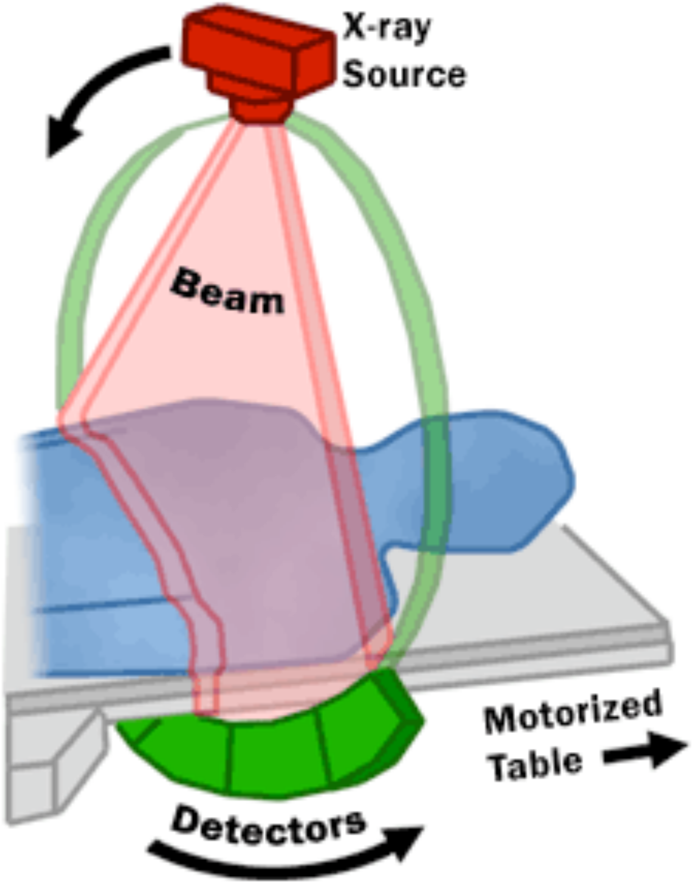
**Direct problem:**

**If the inner structure of a person is known, what would X-ray images of her look like?**

**Inverse problem:**

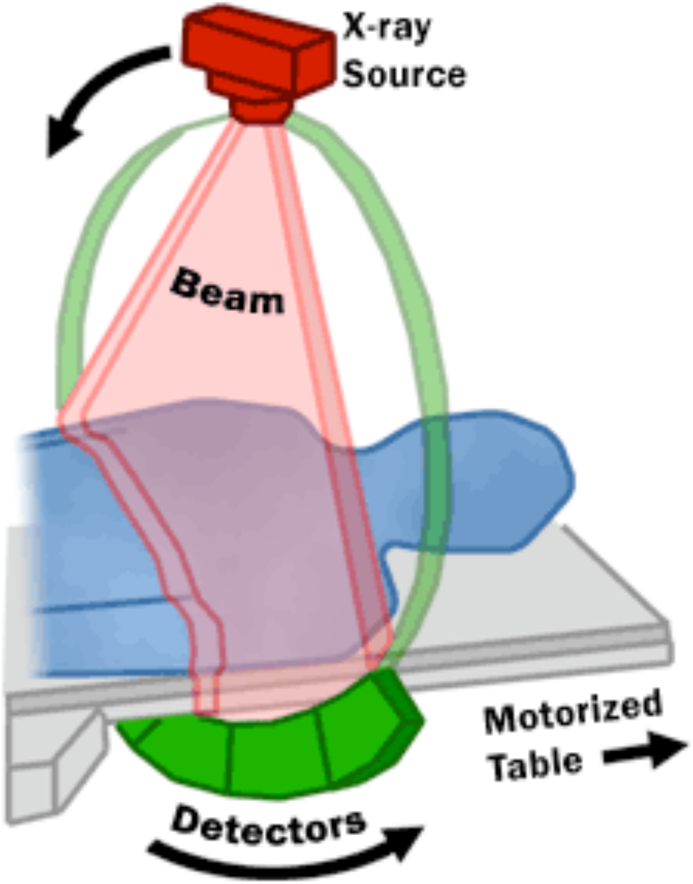
**Given X-ray images from all around the body, what is the inner 3-D structure?**

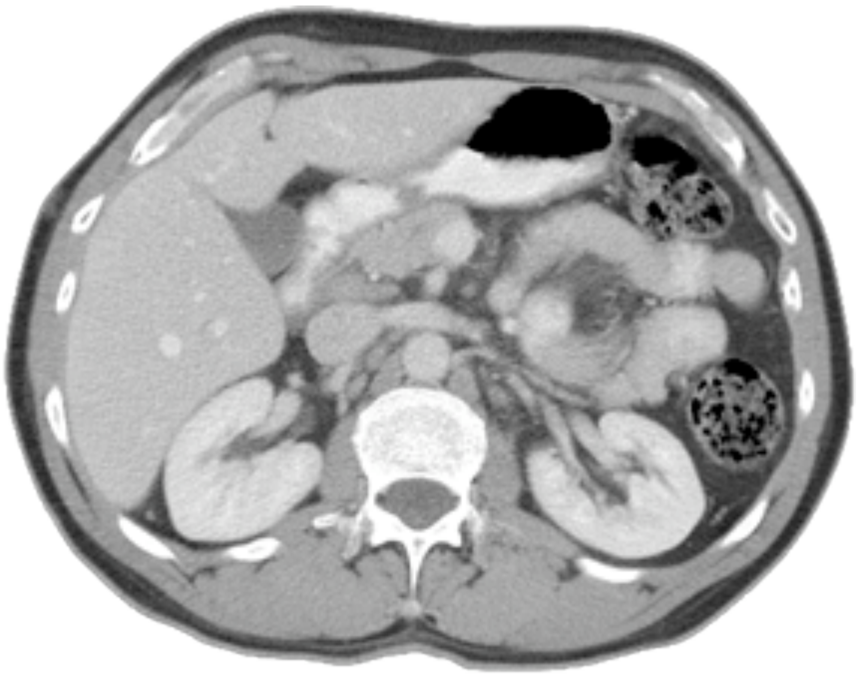
# Traditionally, CT data is collected slice by slice




Images from http://www.fda.gov/cdrh/ct/what.html

# Using a reconstruction algorithm, inner structure in the slice is revealed



**Johann Radon** (1887-1956)

$$f(x) = \frac{1}{4\pi^2} \int_{S^1} \int_{\mathcal{R}} \frac{\frac{d}{ds}(Rf)(\theta, s)}{x \cdot \theta - s} ds d\theta$$

## Filtered back-projection

$$f(x) = \frac{1}{4\pi^2}\int_{S^1}\int_{\mathcal{R}} \frac{\frac{d}{ds}(Rf)(\theta,s)}{x\cdot\theta - s}\,ds\,d\theta$$

Filtered back-projection (FBP) is mathematical technology used on a daily basis in hospitals around the world.

The quality of 3D reconstruction using FBP is excellent.

Nobel prize was awarded to Hounsfield and Cormack 1979.

**However, a comprehensive data set is mandatory for FBP.**

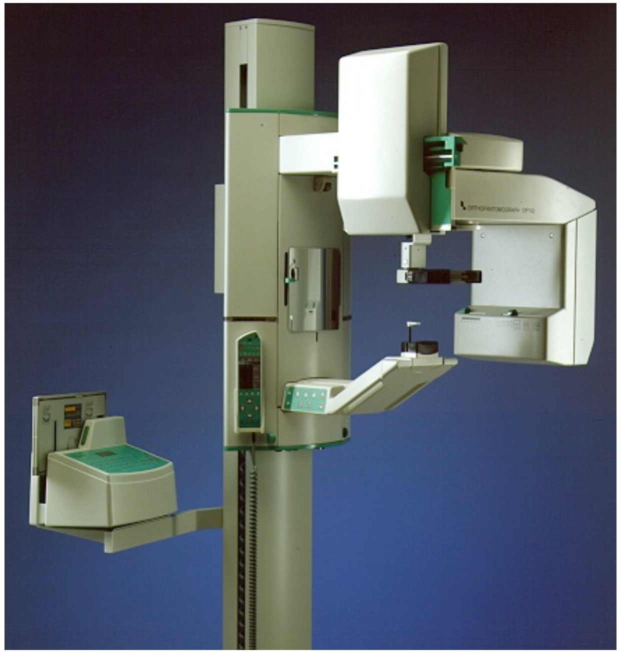
# A series of projects started in 2001 aiming for a new type of low-dose 3D imaging

The goal was a mathematical algorithm with

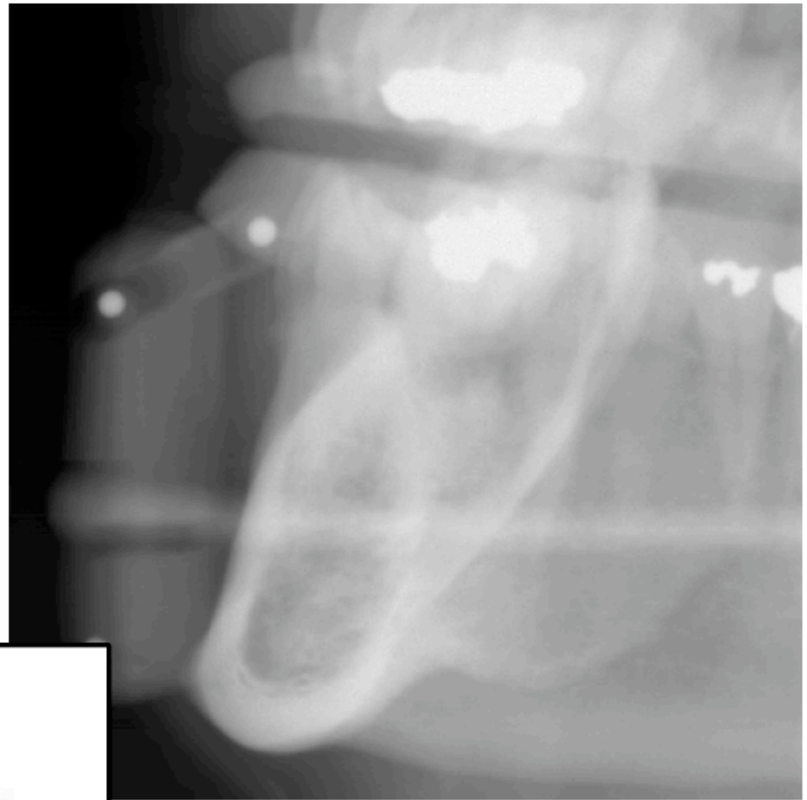
**Input:** small number of digital X-ray images taken with any X-ray device

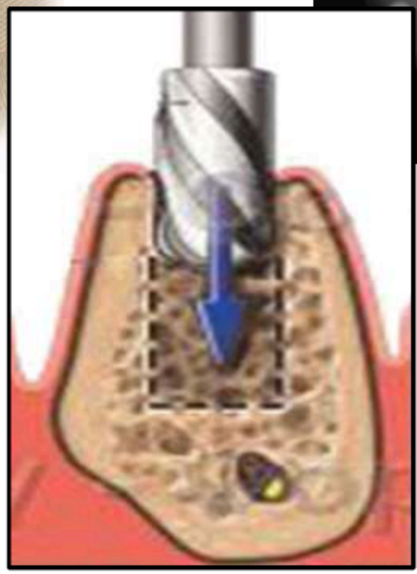
**Output:** three-dimensional reconstruction with quality good enough for the clinical task at hand

Products of Instrumentarium Imaging in 2001:



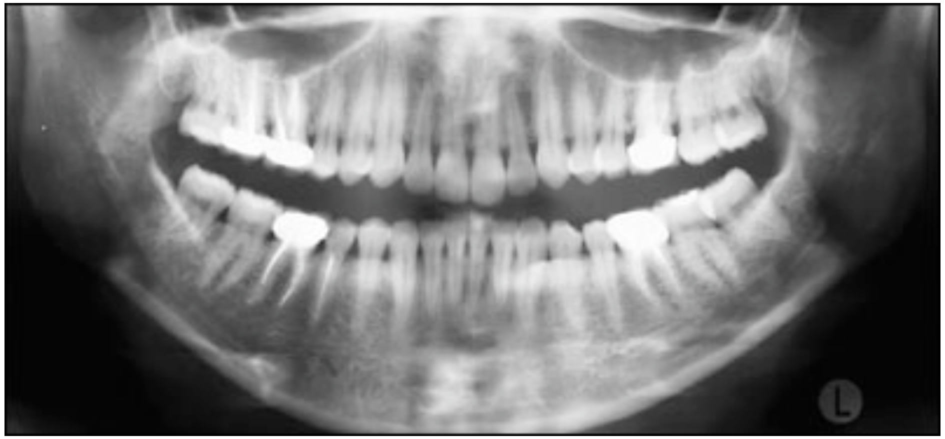
# 2D projection radiograph is not enough for dental implant planning

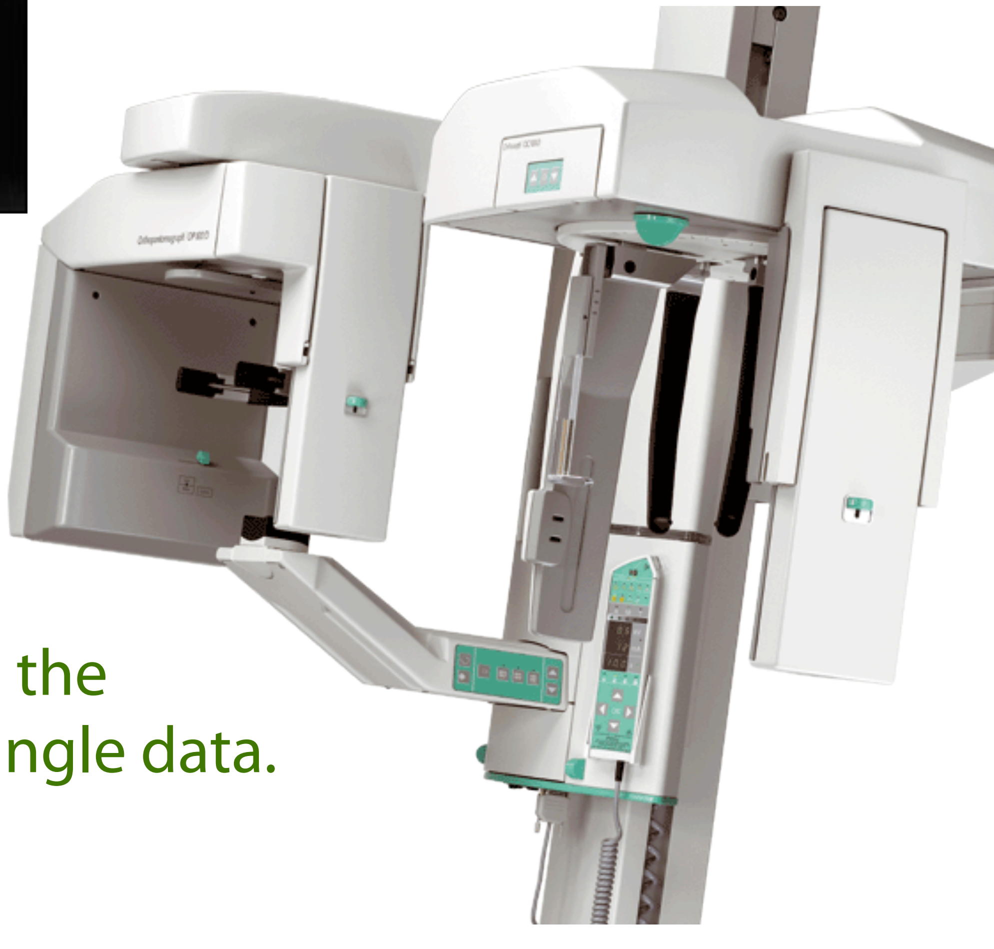





# Panoramic X-ray device rotates around the head and produces a general picture

Panoramic imaging was invented by Yrjö Paatero in 1950's.





Nowadays a panoramic device is standard equipment at every dental clinic around the world.

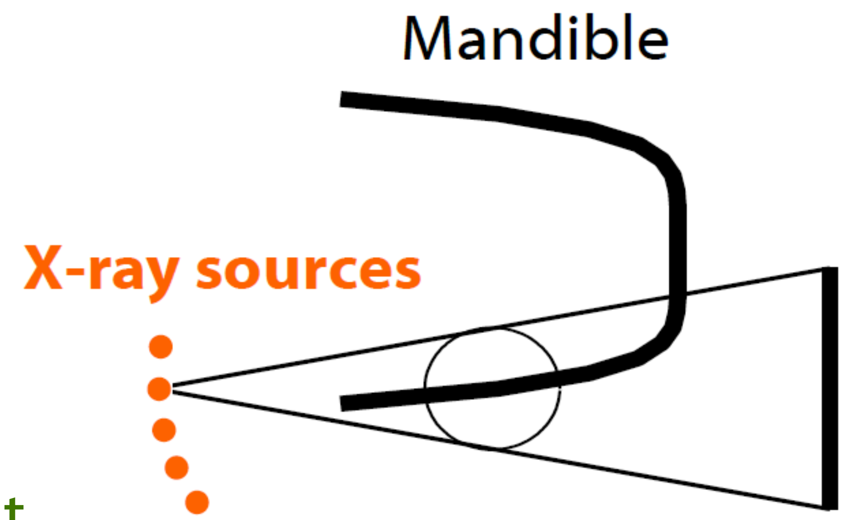
In our project, we reprogrammed the device so that it collects limited-angle data.

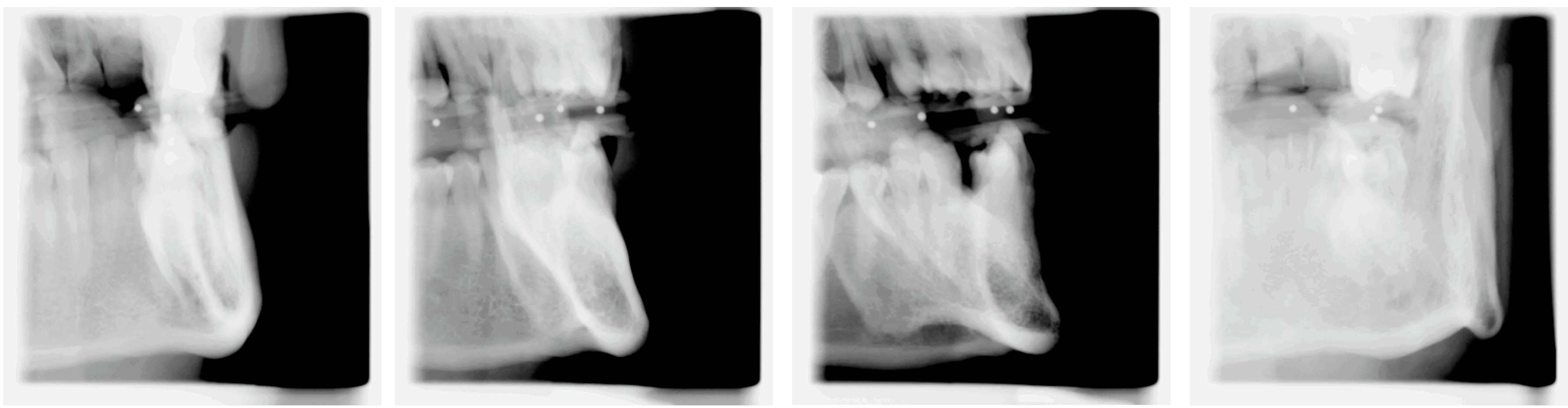
# We consider the following limited angle experiment with the panoramic x-ray device:

**11** projection images of the mandibular area

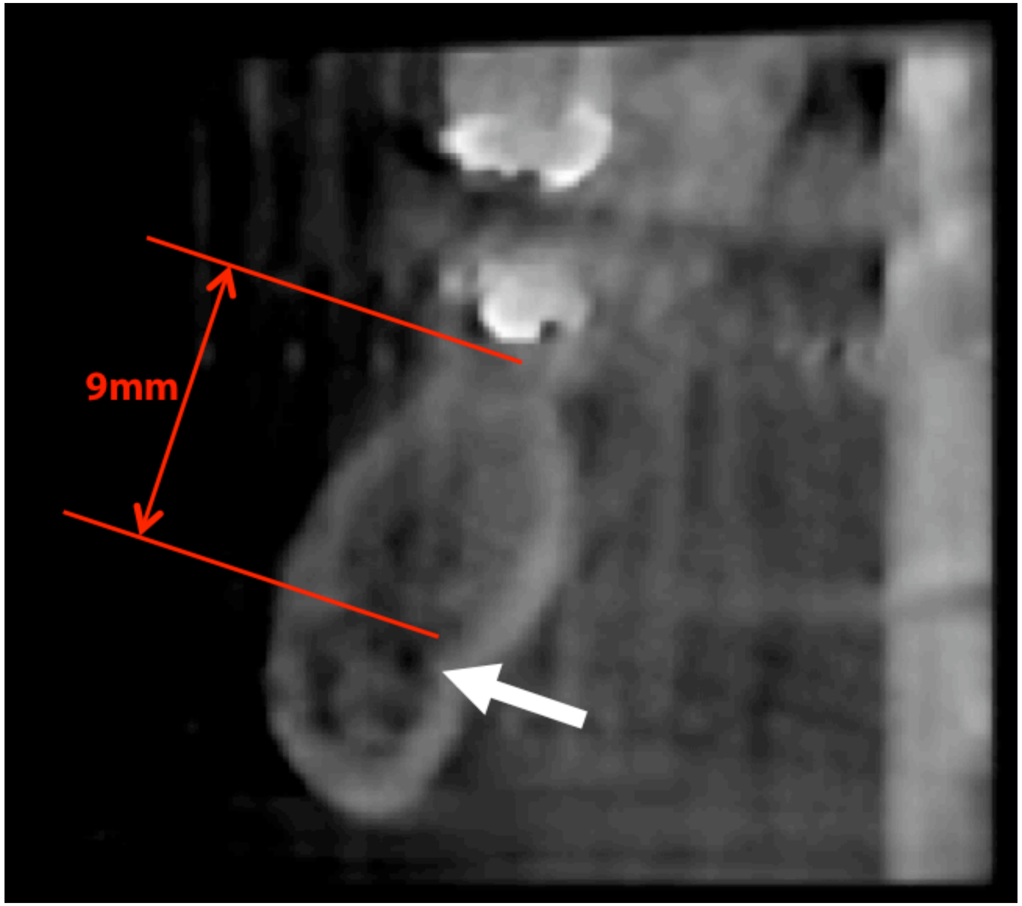
**40** degrees angle of view

**1000 x 1000** pixels per image, formed by a scanning movement



# Limited angle level set reconstruction can be used for locating the mandibular nerve



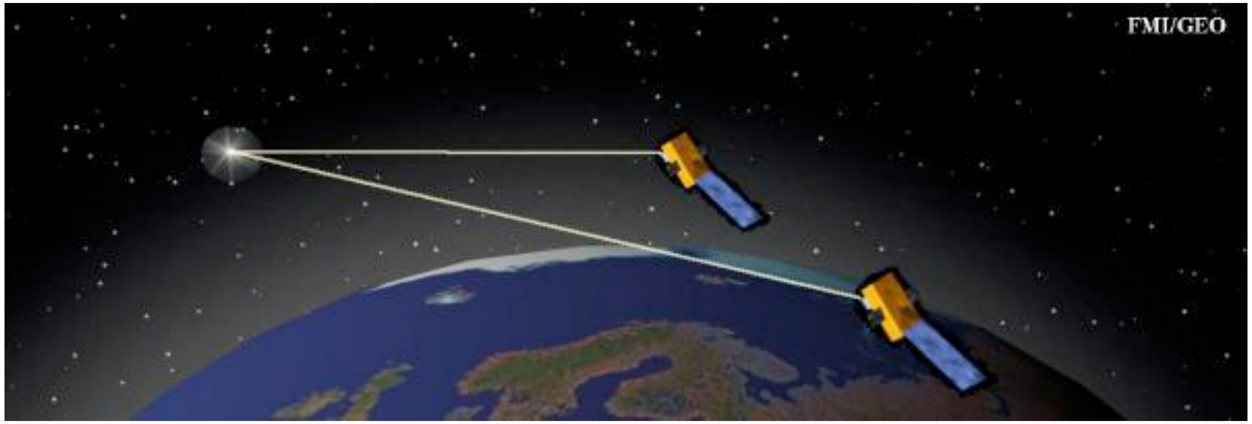

PALODEX GROUP

This is core technology for the PaloDEx Group's VT product that has been in the market since 2007.

Remark that a software update transforms a 2D device into a 3D device.

Kolehmainen, Vanne, S, Järvenpää, Kaipio, Lassas and Kalke (2006)
Kolehmainen, Lassas and S (2008)
Cederlund, Kalke and Welander (2009)
Hyvönen, Kalke, Lassas, Setälä and S (2010)

# Inverse problem: ozone layer tomography




**Direct problem:**

**If the ozone profile of the atmosphere were known, what star occultation measurements would we get?**

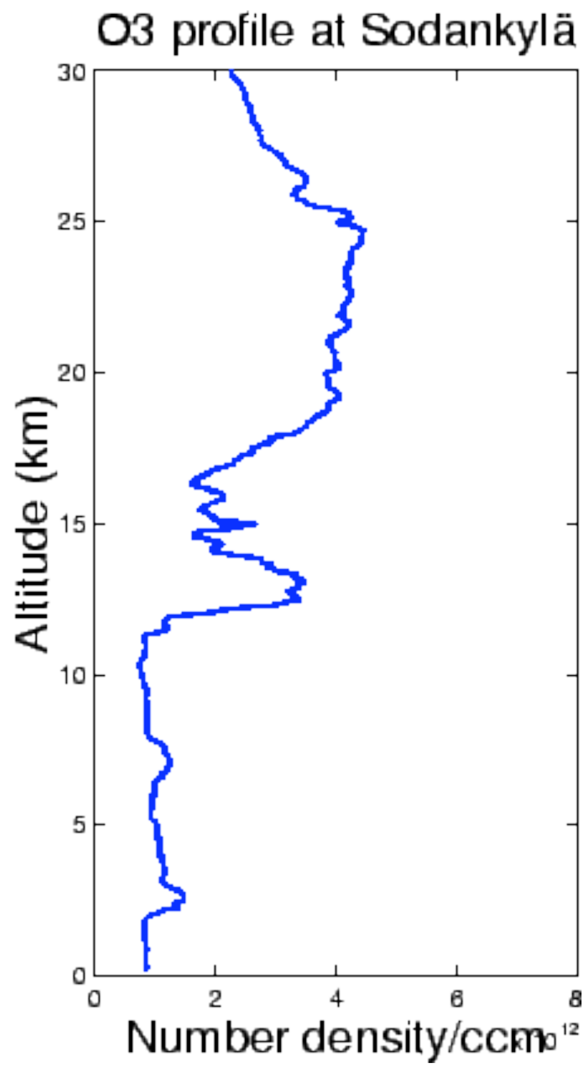
**Inverse problem:**

**Given star occultation measurements , what is the ozone profile?**

# Show animation of measurement!

http://envisat.esa.int/instruments/gomos/descr/flash.html

# As a result we get ozone density as function of altitude




This inverse problem is mathematically the same than the CT problem, except with limited data

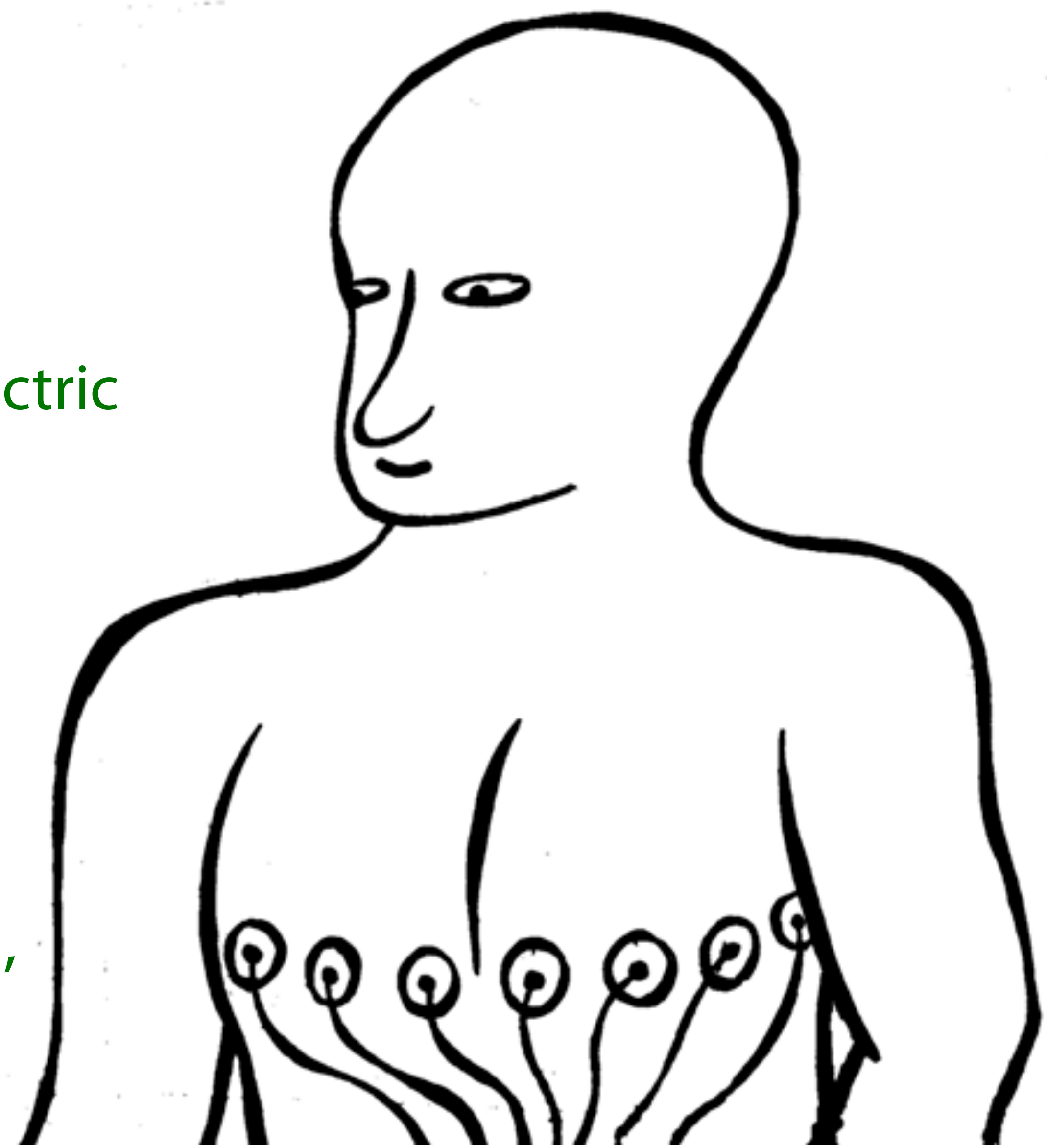
Sources:
European Space Agency
Finnish Meteorological Institute
Envisat and GOMOS projects
http://www.fmi.fi/tutkimus_otsoni/otsoni_26.html
http://envisat.esa.int/handbooks/gomos/CNTR2.htm

# Fourth example of inverse problems: Electrical impedance tomography

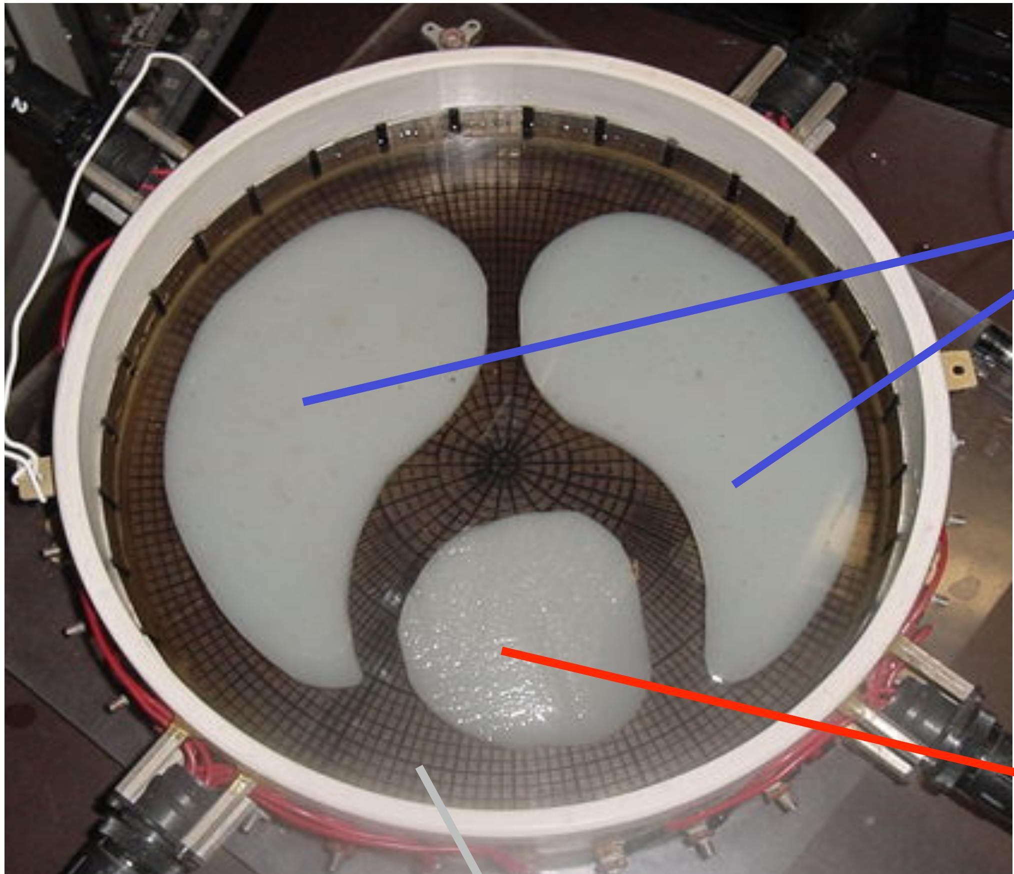
Feed electric currents
through electrodes,
measure voltages

Reconstruct the image of electric
conductivity
in a two-dimensional slice

Applications:
monitoring heart and lungs
of unconscious patients,
detecting pulmonary edema,
enhancing ECG and EEG



# At the RPI lab, we construct a chest phantom consisting of saline and agar

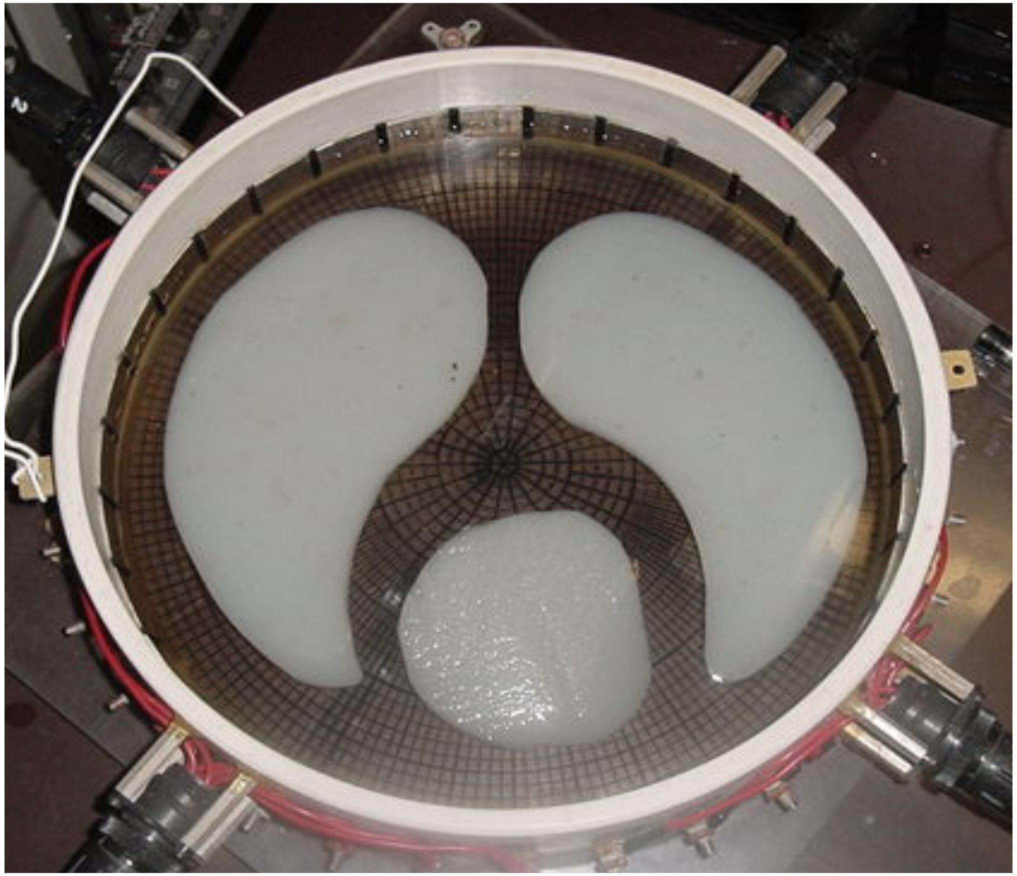


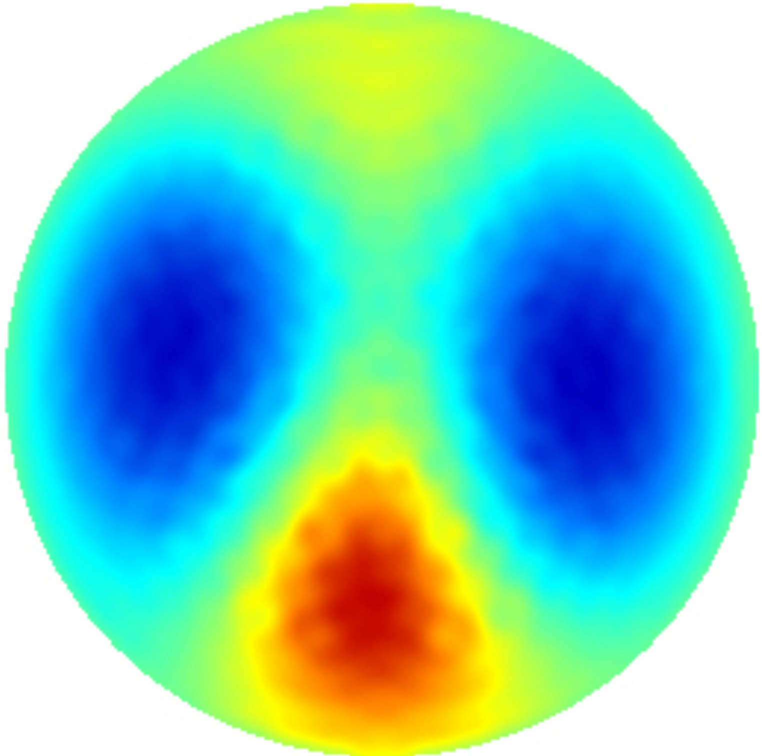
"Lungs" with lower conductivity than background (240 mS/m)

"Heart" with higher conductivity than background (750 mS/m)

Background of salt water, conductivity 424 mS/m.
Diameter of the tank is 30cm.

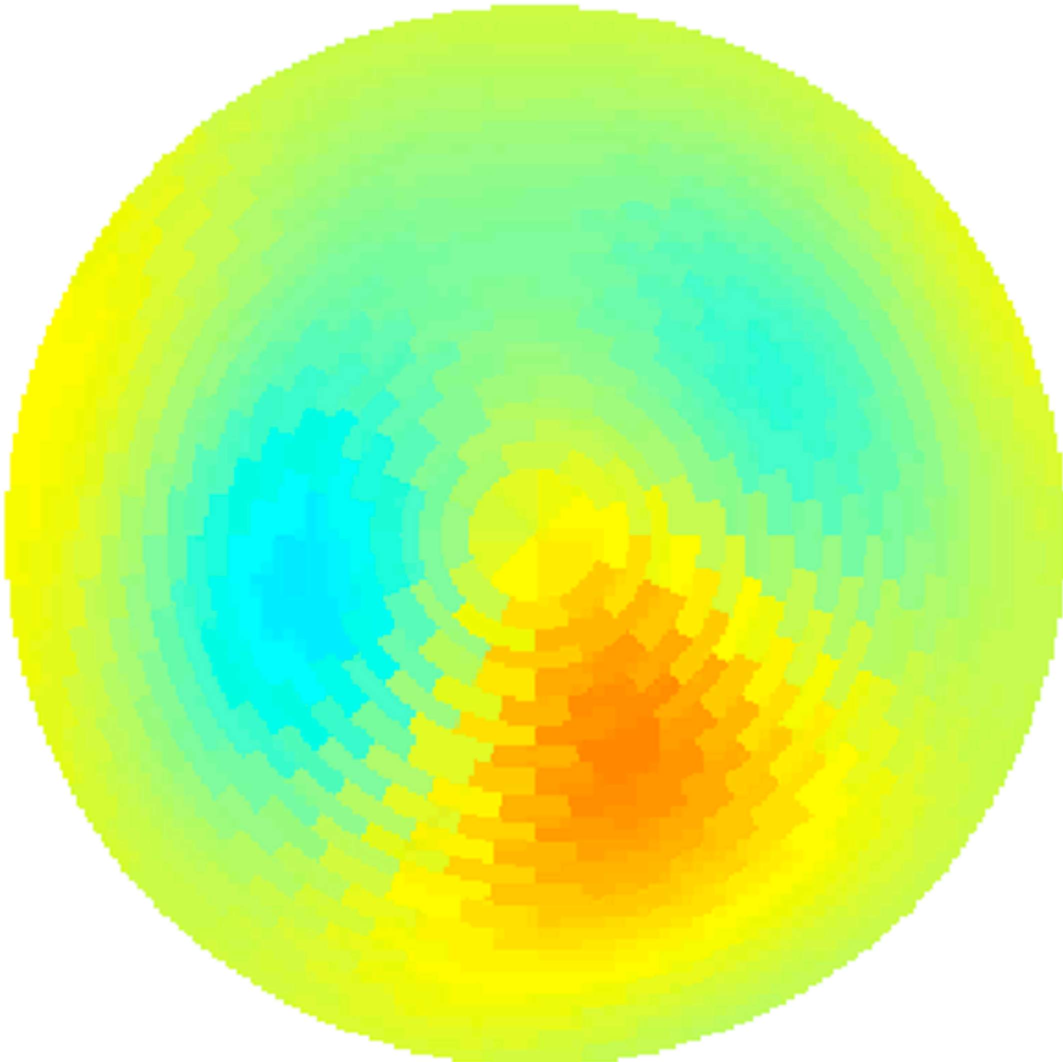
# Reconstruction from phantom data



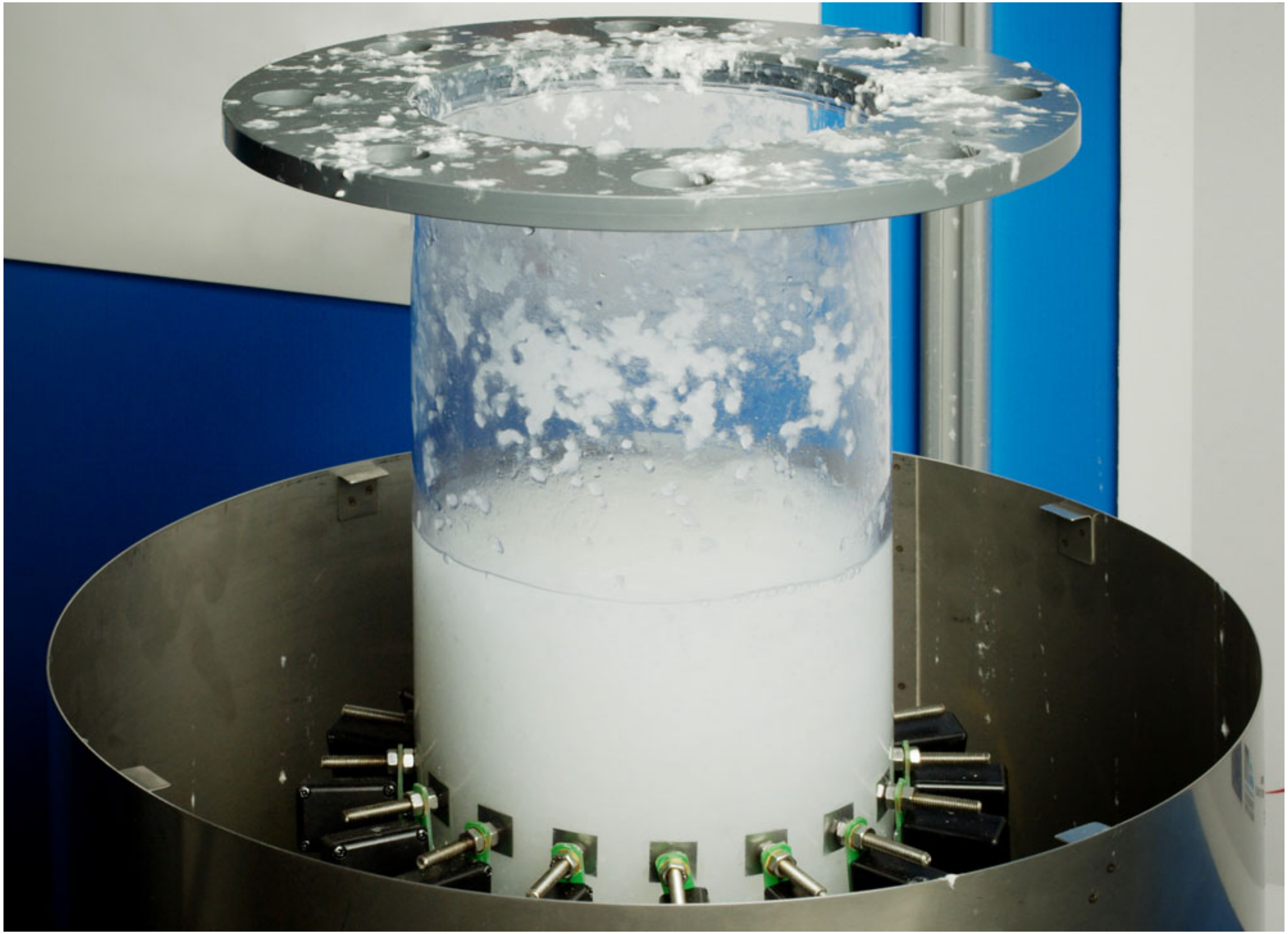


This example is from
Isaacson, Mueller, Newell and Siltanen 2004
IEEE Transactions on Medical Imaging 23, pp. 821-828

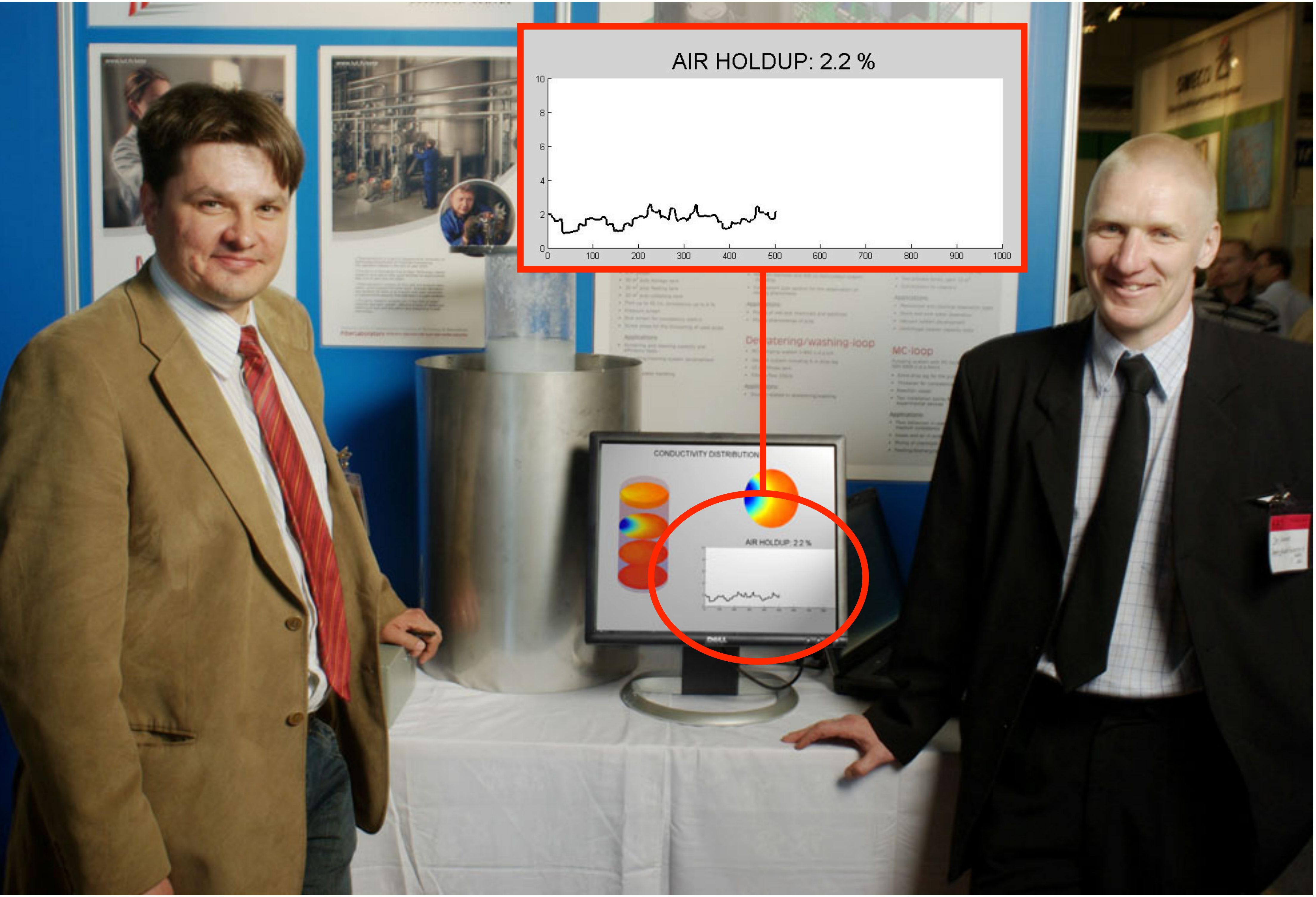
# Reconstruction from data collected from a living person



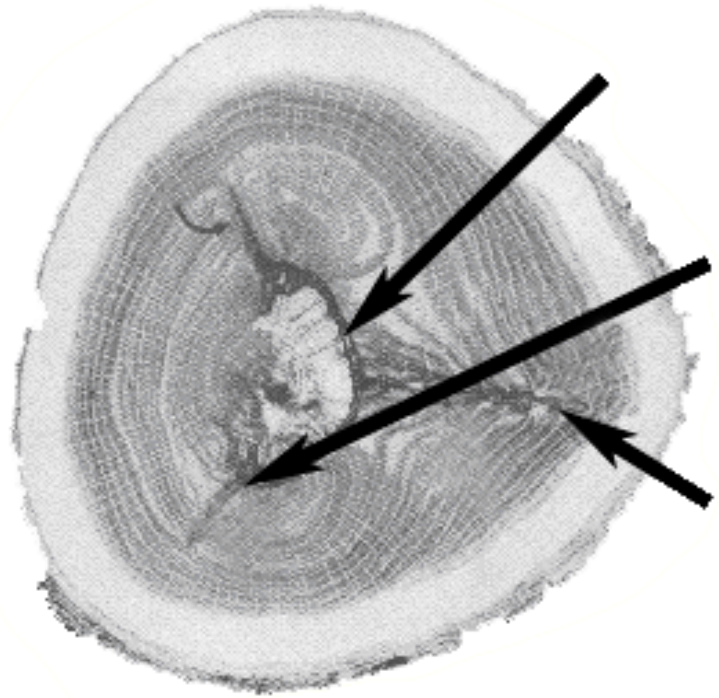
# EIT can be used as well in industrial process monitoring




**Lasse Heikkinen and Jari Kourunen**
**University of Kuopio, Finland**

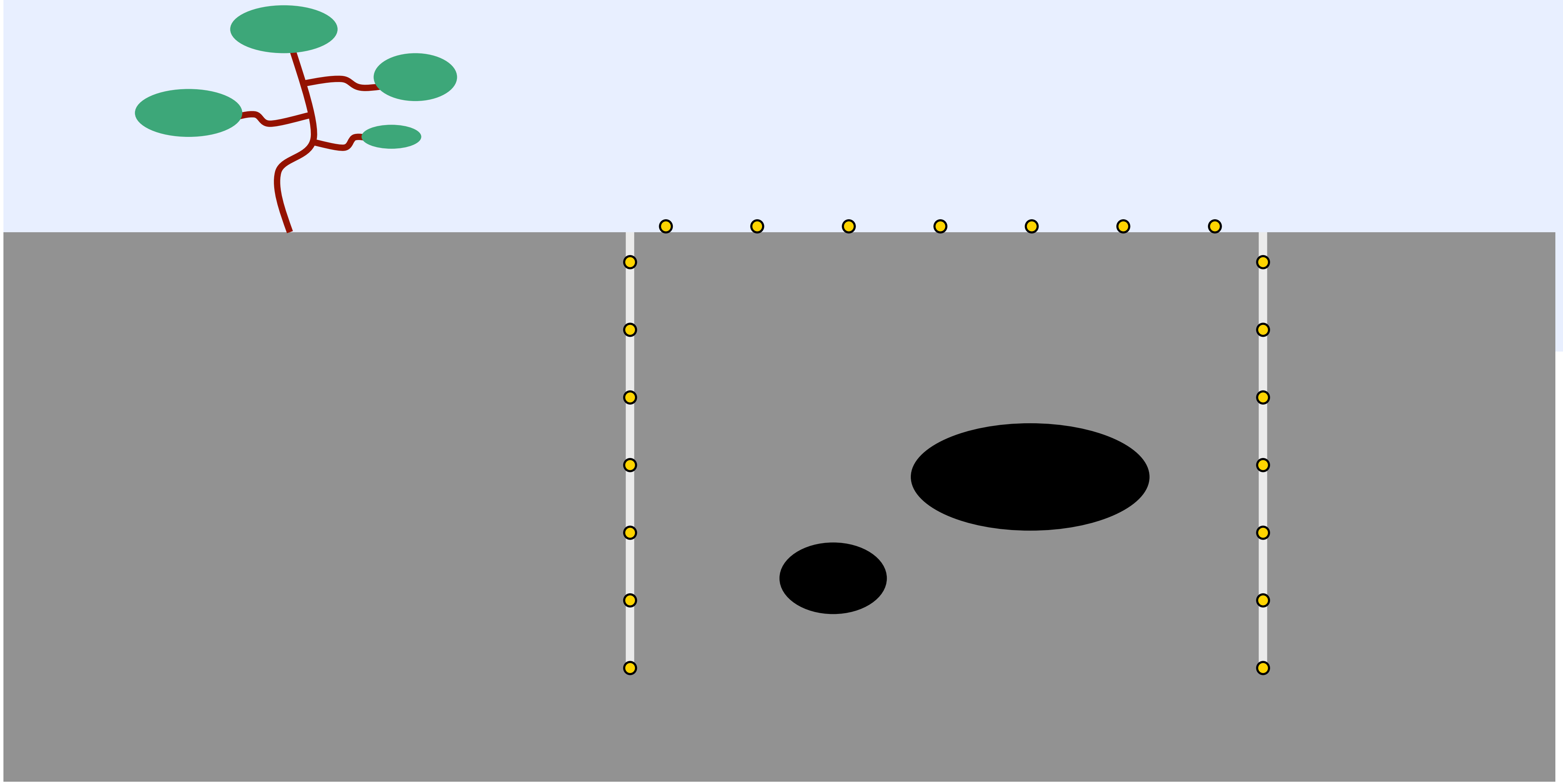

AIR HOLDUP: 2.2 %
10
8
6
4
2
0
0
100
200
300
400
500
600
700
800
900
1000
CONDUCTIVITY DISTRIBUTION
AIR HOLDUP: 2.2 %
MC-loop

# EIT can be used for finding defects in materials

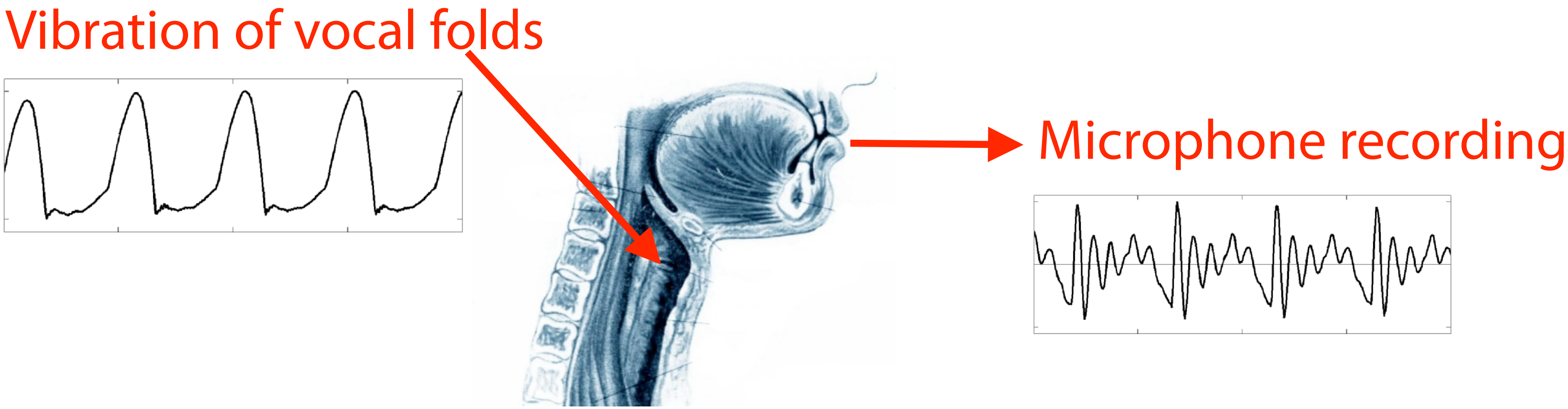


*Shigo, A.L., 1983. Tree Defects: A Photo Guide. USDA Forest Service, No. Cent. For. Exp. Sta., GTRNE-82.*

# Geological sensing of oil or metals is another application of EIT



# Glottal Inverse Filtering (GIF)




**Direct problem:** If we know the vibration signal at the vocal folds and the shape of the vocal tract, what does the microphone record?

**Inverse problem:** Given the microphone recording, find the vibration signal at the vocal folds and the shape of the vocal tract. This is called GIF filtering.

# GIF-filtering has many applications, for example speech synthesis and recognition

Let's compare two synthetic voices

(1) Commercial basic technique,

(2) Novel method developed by Paavo Alku (Aalto University) and Martti Vainio (University of Helsinki).

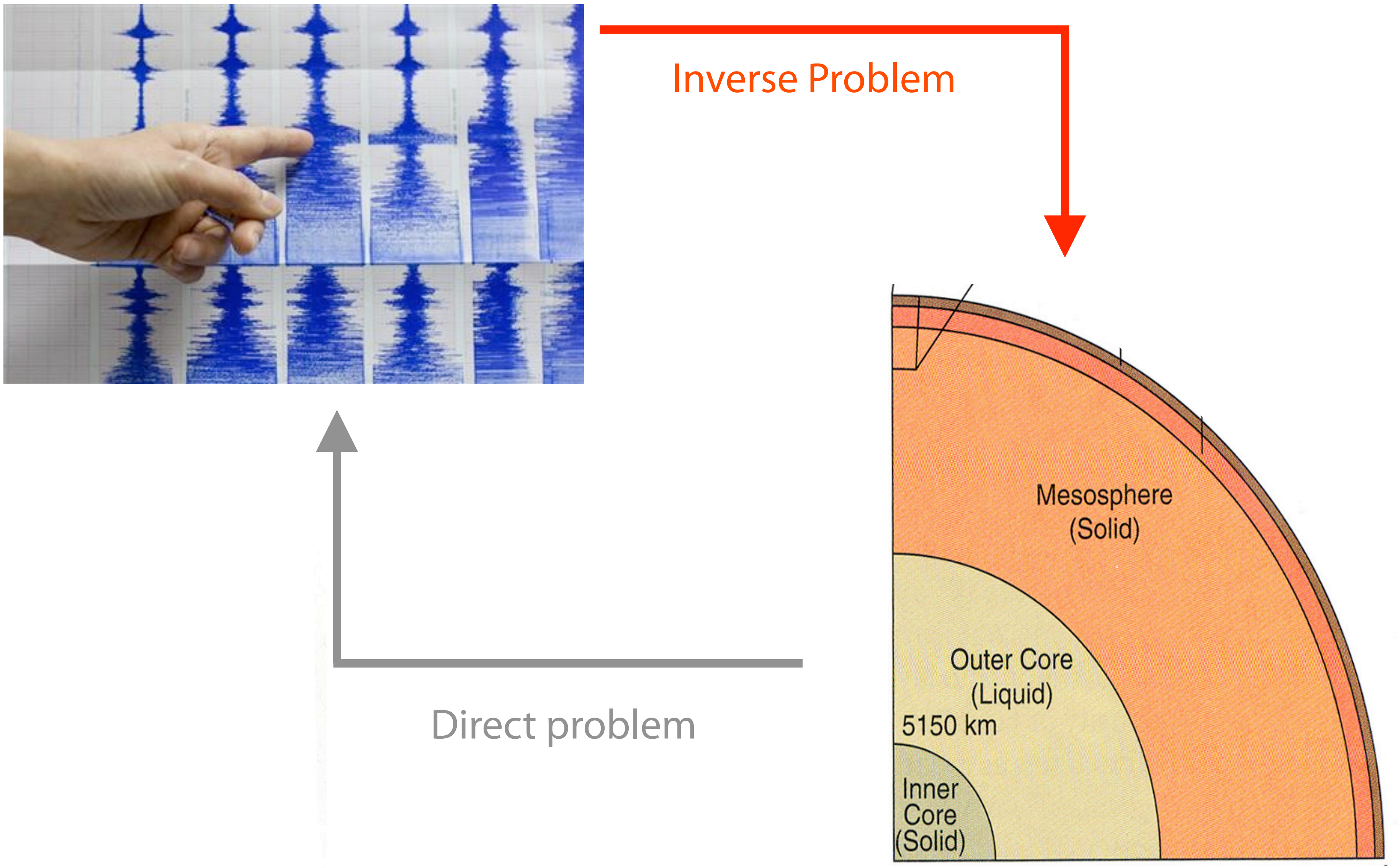
# Sixth example of inverse problems: Recovering the inner structure of Earth

**Direct problem:**

**Given the inner structure of Earth,**
**predict vibrations caused by an earthquake**

**Inverse problem:**

**Given earthquake data around the world,**
**find sound speed distribution inside Earth**




(University of New Hampshire )

# Inverse problem: pricing financial instruments

# The Black-Scholes equation is the mathematical model behind option pricing

Under "perfect market" assumptions
(liquidity, absence of arbitrage and transaction cost)
the call price C satisfies

$$\partial_t C + \frac{1}{2}\sigma^2(t,S)S^2\partial_S^2 C + r(S\partial_S C - C) = 0,$$
$$C(S,T) = (S-K)^+,$$

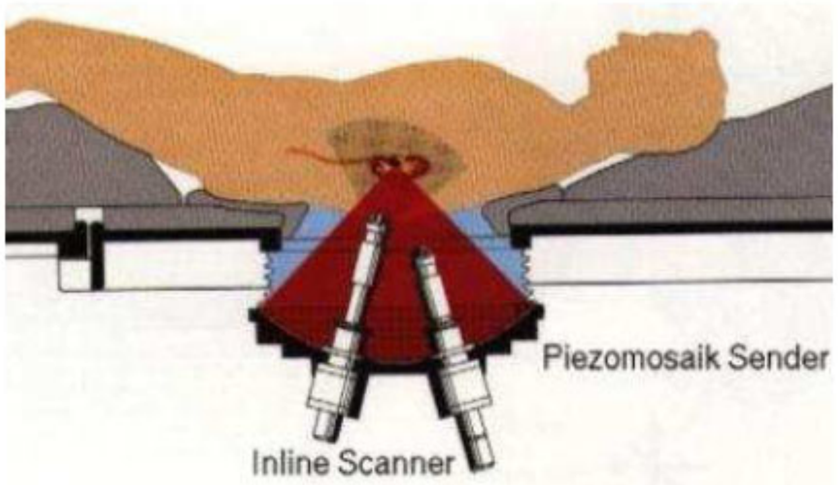
where r is the constant interest rate on a riskless investment.

The inverse problem is to determine the local volatility $\sigma(S,t)$
from a set of noisy call prices:

$$\{C^\delta(S,t;K,T)\}_{(K,T)\in\mathcal{I}}$$

# Fifth example of inverse problems: Ultrasound therapy

Crushing kidney stones
using ultrasound
(Riedlinger, Karlsruhe)

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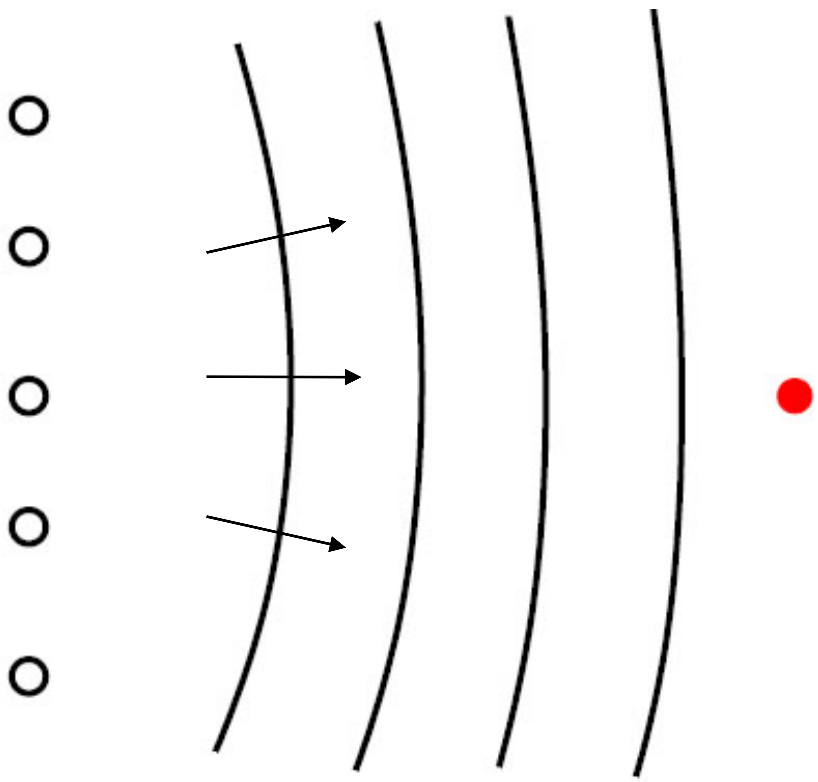
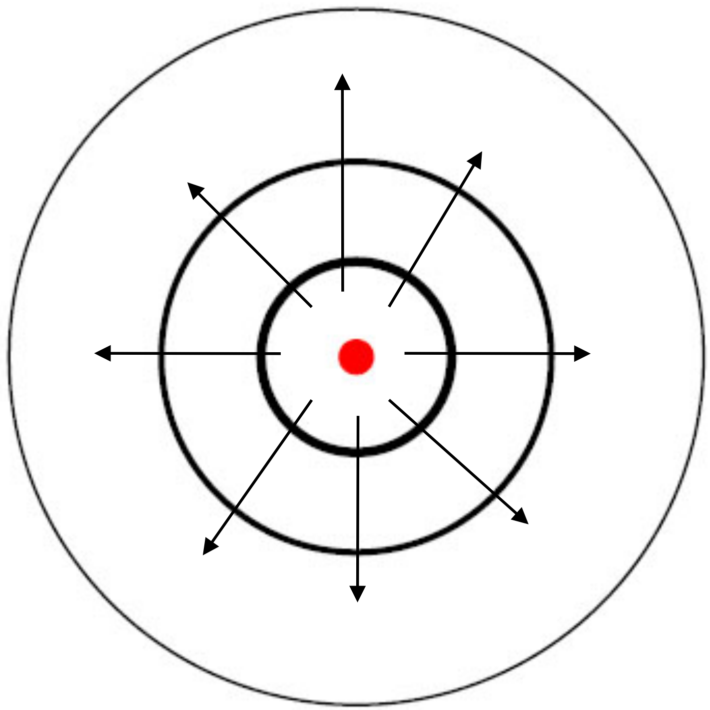
Focusing wave
(Kaipio, Kuopio)

# One possibility to focus waves is *time reversal*

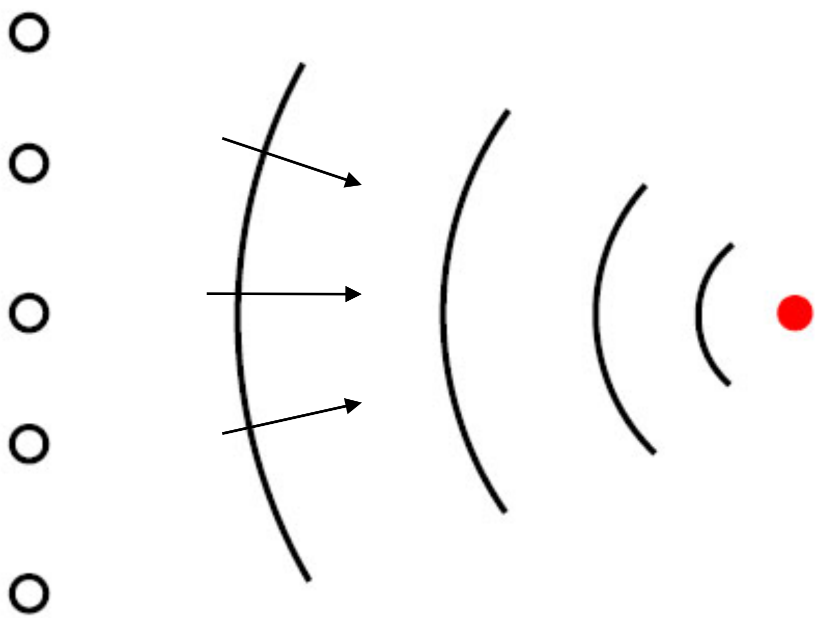
1. Send a probing signal



1. Send a probing signal
2. Measure and record the response

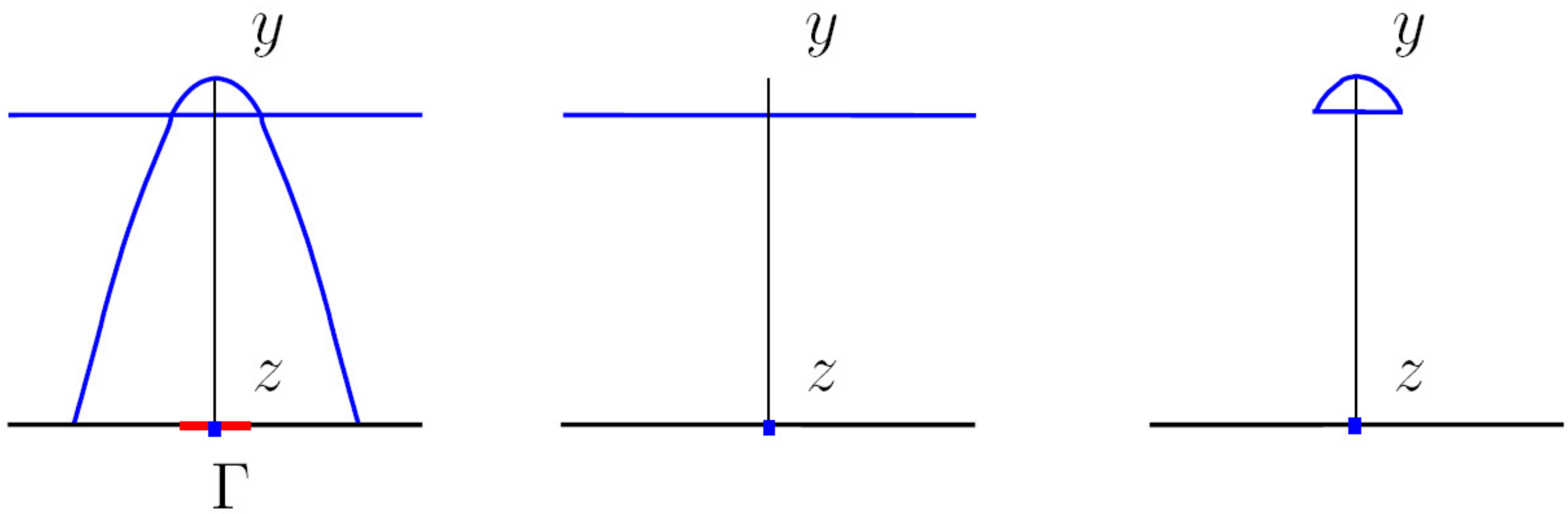


1. Send a probing signal
2. Measure and record the response
3. Send response back reversed in time



# Time-reversal based focusing can recover sound speed inside unknown bodies

$$
\begin{aligned}
F &:= \alpha P(J\Lambda_{2T} - R\Lambda_{2T}RJ)f, \\
a_n &:= \Lambda_{2T}(h_n), \\
b_n &:= \Lambda_{2T}(RJh_n), \\
h_{n+1} &:= (1-\alpha^2)h_n + \alpha(PRb_n - PJa_n) + F
\end{aligned}
$$

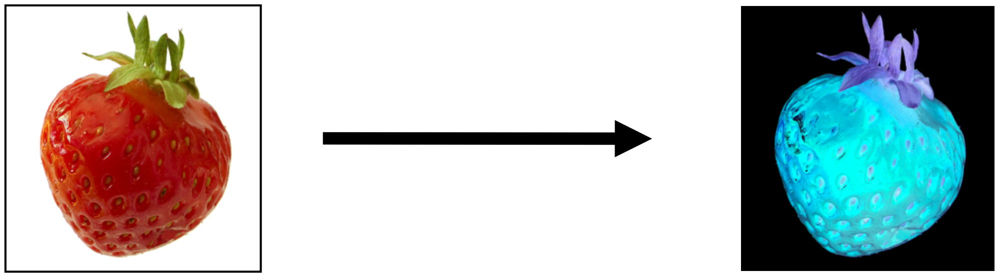


# What are *not* inverse problems?

# Example of a non-inverse problem: Inverting a photograph

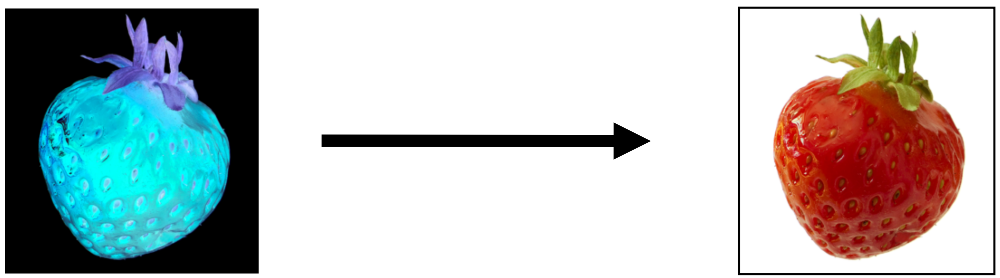
**Direct problem:**

**Given a photograph,**
**determine the negative image**



**"Inverse problem":**

**Given a negative,**
**determine the positive image**



# Hadamard's definition of a "well-posed problem" has three parts


**(H1) A solution exists**

**(H2) The solution is unique**

**(H3) The output depends continuously on the input**

**A problem is called "ill-posed", or inverse problem, if (H1), (H2) or (H3) fails.**

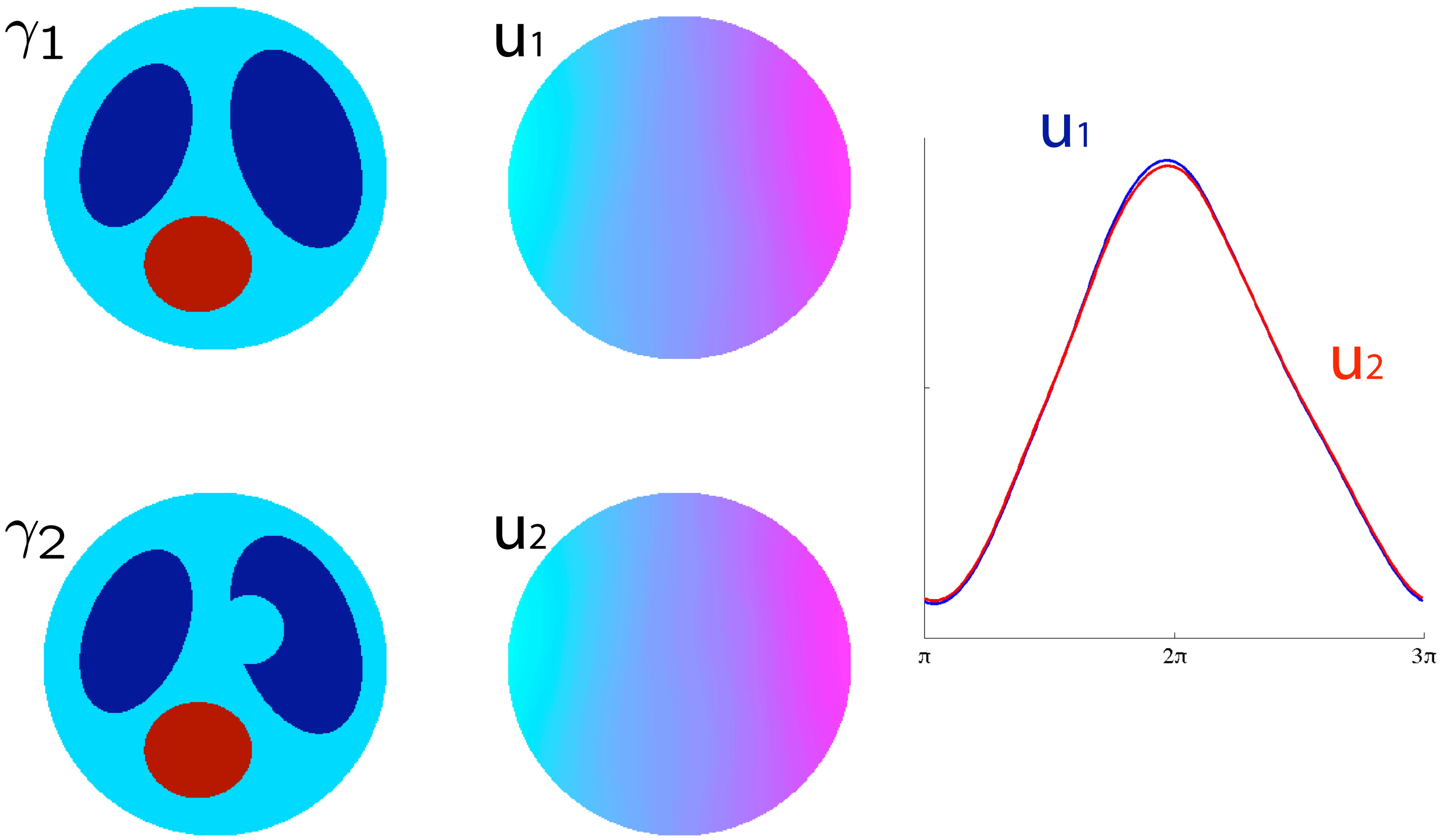
Jacques Salomon Hadamard (1865-1963)

# Example inverse problems revisited

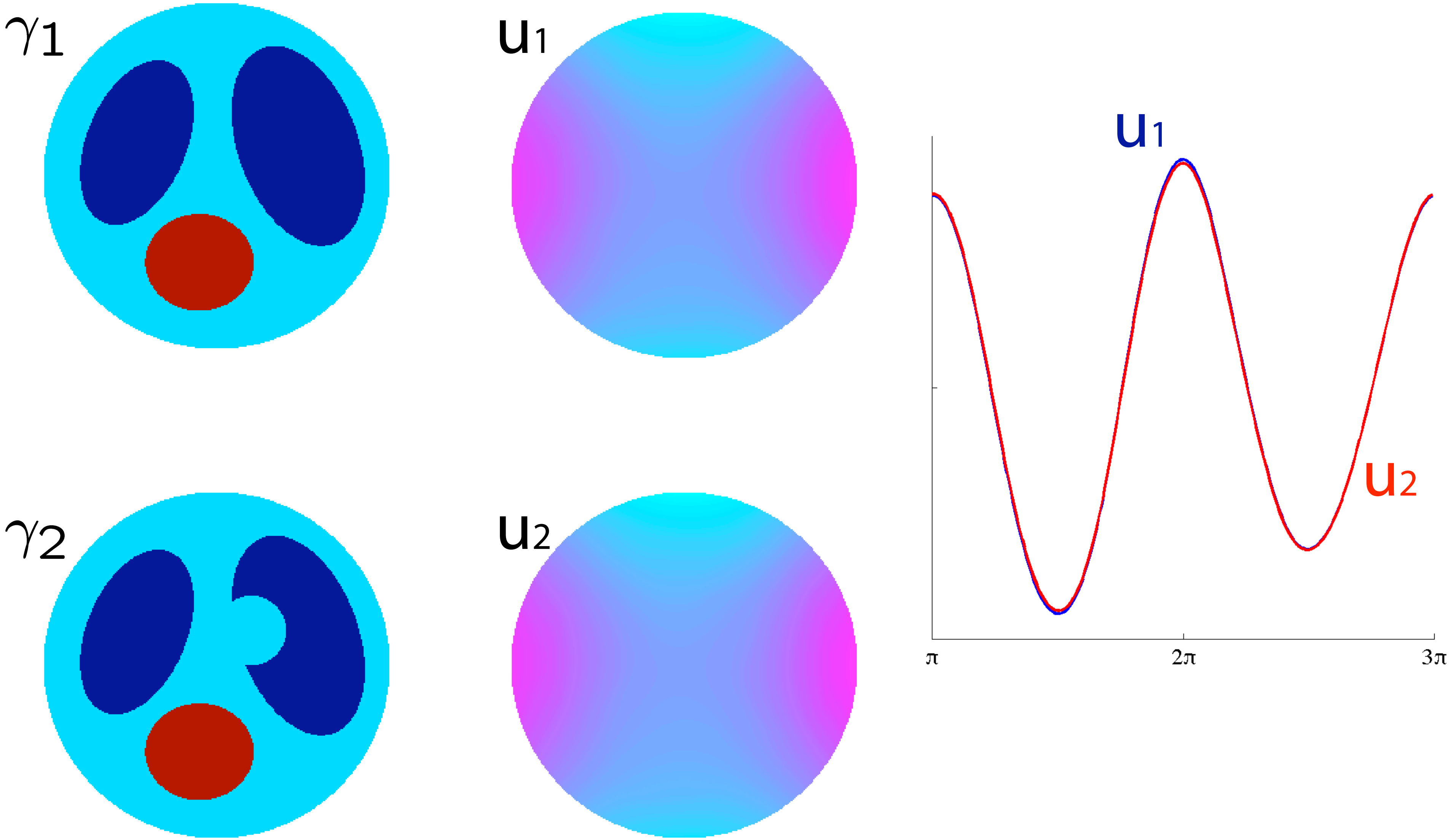
# Image deblurring

**Changing few pixel values on the left**
**changes the blurred image only slightly: (H3) fails**

# Electrical impedance tomography: choose two different conductivities



# Electrical impedance tomography: big change in conductivity causes only small change in data



# What is this course all about?

# Goals of the course:

1. Learn how to write a practical inverse problem in matrix form: $m = Af + \varepsilon$
2. Learn how to detect ill-posedness from a matrix A using Singular Value Decomposition
3. Familiarize with two classes of solution methods: regularization and statistical inversion
4. Acquire skills to solve practical inverse problems using Matlab
5. Learn to report your scientific findings in writing

# Good to know

Matrix algebra

Least squares solution of linear systems

Basic Matlab programming (do you have access to Matlab and Image Processing Toolbox?)

Basic probability

# How to pass the course?

Return solutions to exercise problems

Pass final exam

Complete project work