Introduction to differential forms
model solutions 8

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1.

solution. 1. Recall that

$$\begin{aligned} S_1\omega(x) &= \int_{[0,1]} \partial_t \llcorner F^*\omega_{t,x}\, dt \\ &= \int_{[0,1]} \tilde{\gamma}_x^*\omega \\ &= \int_{\tilde{\gamma}_x} \omega. \end{aligned}$$

Where $\tilde{\gamma}_x : [0,1] \to U \ t \mapsto x_0 + \lambda(t)(x - x_0)$. By change of variables this is equal to

$$\int_{\gamma_x} \omega.$$

□

2.

solution. We define the map $\tilde{H}\omega(x) = \hat{S}_k \circ H^* = \int_0^1 \partial_t \llcorner H^*\omega_{(x,t)}\, dt$. Then we have from lectures that

$$d\hat{S}_k H^*\omega + \hat{S}_k dH^*\omega = d\hat{S}_k H^*\omega + \hat{S}_k H^* d\omega = \partial_t H^*\omega_{(x,1)} - H^*\omega_{x,0} = g^*\omega - f^*\omega.$$

If $d\omega = 0$, then $g^*\omega - f^*\omega = d\hat{S}_k H^*\omega$, hence their difference is exact, and the map $g^* - f^*$ is zero on cohomology so the two maps are equal. □

3.

solution. Consider $g^*f^*[\alpha] = g^*[f(\alpha)] = [g(f(\alpha))] = 0$ by the exactness of

$$A \xrightarrow{f} B \xrightarrow{g} C.$$

□

4.

solution. We can explicitly construct the isomorphism. Choose $[\alpha] \in H^k(U \cup V)$, and let $\Phi([\alpha]) = ([\xi_U \alpha], [\xi_V \alpha])$. This is well defined because $d\xi_U = d\xi_V = 0$, so

$$\xi_U(\alpha + d\tau) = \xi_U \alpha + \xi_U d\tau = \xi_U \alpha + d\xi_U \tau.$$

Hence Φ is well defined. The homomorphism Φ has an inverse given by

$$([\alpha], [\beta]) \mapsto [E_U(\alpha) + E_V(\beta)],$$

where E_U is extension by 0 outside of U. Once again this is well defined because $E_U(d\tau) = d(E_U \tau)$. □

5.

solution.

1. Let e_i $i \in I$, be a basis of A and $J \subset I$ make $\{e_j : j \in J\}$ a basis of $\ker f$. Then we consider the map $\sum_{i \in I \setminus J} v_i e_i + \sum_{j \in J} v_j e_j \mapsto \sum_{i \in I \setminus J} v_i f(e_i) \oplus \sum_{j \in J} v_j e_j$. But this is invertible because if $\sum_{i \in I \setminus J} v_i f(e_i) = 0$ then

   $$f\left(\sum_{i \in I \setminus J} v_i e_i\right) = 0,$$

   and

   $$\sum_{i \in I \setminus J} v_i e_i \in \ker f,$$

   which is a contradiction to the choice of basis.

2. We know that $B \cong \ker g \oplus \mathrm{Im} g$, but by the exactness of the sequence $\mathrm{Im} g = \ker 0 = C$, and $\ker g = \mathrm{Im}\, f$. But $\ker f = 0$, so $\mathrm{Im}\, f \cong A$ so $B \cong A \oplus C$.

□

6.

solution. The five lemma is a classical result in homological algebra. Its proof is the quintessential example of "diagram chasing". First we want to show that f_3 is injective if f_1 is surjective, and f_2 and f_4 are injective.

Suppose $f_3(a) = 0$. By commutativity f_4 of the image of a is 0 so by injectivity the image of a is 0, so $a \in \ker A_3 \to A_4$ then the is an $a' \in A_2$ that maps to a. Then $f_2(a') \in \ker B_2 \to B_3$ by commutativity of the diagram, hence there is a $b \in B_1$ that maps to $f_2(a')$. But f_1 is surjective so there is an a'' such that $f_1(a'') = b$. By the injectivity of f_2 and commutativity we know that a'' maps to a', but then a' in in the image of $A_1 \to A_2$ and hence in the kernel of $A_2 \to A_3$, so a (the image of a') is 0. Hence f_3 is injective.

If one were a category theorist one could immediately conclude that the dual result holds for the dual diagram (arrows reversed), which would prove the result... we are not category theorists.

Choose $b \in B_3$. Let b' denote the image of b, then there is an a' such that $f_4(a') = b'$. By exactness b' is in the kernel of $B_4 \to B_5$, and by commutativity f_5 of the image of a' is 0, but by the injectivity of f_5 the image of a' is 0, so a' is in $\ker A_4 \to A_5$, and there is an a that maps to a'. Then by commutativity $b - f_3(a) \in \ker B_3 t \to B_4$, therefore there is a b'' that maps to $b - f_3(a)$, but by the surjectivity of f_2 there is an a'' for which $f_2(a'') = b''$, then by commutativity the image of a'' plus a maps under f_3 to b. □