Introduction to differential forms

model solutions 7

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1.

solution. 1. The easiest way to construct a chain is to start with a the boundary of a simple 3-chain, and map it to S^2. Let ζ be the 3-simplex in $\mathbb{R}^3$ given by $[A, B, C, D]$ where

$$A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad B = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} \quad C = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ -\frac{\sqrt{3}}{4} \end{pmatrix} \quad D = \begin{pmatrix} -\frac{\sqrt{3}}{4} \\ -\frac{3}{4} \\ -\frac{1}{2} \end{pmatrix}.$$

Now let $\pi : \mathbb{R}^3 \setminus \{0\} \to S^2$ be the map $x/|x|$. Then by virtue of being the boundary of a convex domain $|\partial\zeta|$ is mapped homeomorphically to S^2, hence we can take $\pi(\partial\zeta)$ as an appropriate chain.

2. Now we must examine $\partial\pi(\partial\zeta)$. We note that $\partial[A, B, C, D] = -[A, B, C] + [A, B, D] - [A, C, D] + [B, C, D]$. So

$$\pi \circ \partial\zeta = -\pi \circ [A, B, C] + \pi \circ [A, B, D] - \pi \circ [A, C, D] + \pi \circ [B, C, D],$$

and

$$\begin{aligned} \partial\pi \circ \partial\zeta &= -\pi \circ [B, C] + \pi \circ [A.C] - \pi \circ [A.B] + \pi \circ [B, D] \\ &\quad - \pi \circ [A, D] + \pi \circ [A, B] - \pi \circ [C, D] + \pi \circ [A, D] \\ &\quad - \pi \circ [A, C] + \pi \circ [C, D] - \pi \circ [B, D] + \pi \circ [B, C] \\ &= 0 \end{aligned}$$

hence it is a cycle.

3. In fact it is the boundary of the 3-simplex given by the map $\Pi : x \to (d(0, \partial\zeta) - d(x, \partial\zeta))\sigma$, composed with sigma.

□

2.

solution1. Let $\omega = x\,dy\wedge dz + y\,dx\wedge dz + z\,dx\wedge dy$. Let $\sigma = d\tau$ where τ is a diffeomorphism of standard three-simplex to $B(0,1)$. Then

$$\int_\sigma \omega = \int_{d\tau} \omega = \int_\tau d\omega = \int_{B(0,1)} dx\wedge dy\wedge dz = \frac{4}{3}\pi r^3.$$

Note that σ is a cycle in $\mathbb{R}^3\backslash\{0\}$ even if it is not a boundary. But it is a boundary in $\mathbb{R}^3$. The form ω extends smoothly to all of $\mathbb{R}^3$, so there is no awkwardness with this argument. $\square$

solution2. Consider the heuristic 2-form $\omega = r\sin\theta d\theta\wedge d\phi$. This is the spherical surface form, with the functions

$$\theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \qquad \phi = \arctan\left(\frac{y}{x}\right),$$

wherever they need to be defined. If we expand these out , we arrive at

$$\omega = \frac{1}{r^2}\left(z\,dx\wedge dy - y\,dx\wedge dz + x\,dy\wedge dz\right).$$

Now we examine the integral

$$\int_{\pi\circ[B,C,D]} \omega = \int_{[B,C,D]} \pi^*\omega.$$

If we set $\Delta = [B', C', D']$ where $B' = (\sqrt{3}/2, 0)^t$, $C' = (\sqrt{3}/4, 3/4)^t$ and $D' = (\sqrt{3}/4, 3/4)^t$, and

$$\sigma(w) = \sqrt{|w|^2+1/4}^{-1}\begin{pmatrix} w^1 \\ w^2 \\ 1/2. \end{pmatrix}$$

Then

$$\begin{aligned}
\sigma^* dx &= \frac{dw^1}{\sqrt{|w|^2+1/4}} - \frac{w^1}{|w|^2+1/4}\frac{w^1dw^1+w^2dw^2}{\sqrt{|w|^2+1/4}} \\
\sigma^* dy &= \frac{dw^2}{\sqrt{|w|^2+1/4}} - \frac{w^2}{|w|^2+1/4}\frac{w^1dw^1+w^2dw^2}{\sqrt{|w|^2+1/4}} \\
\sigma^* dz &= -\frac{w^1dw^1+w^2dw^2}{2\sqrt{|w|^2+1/4}^3}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma^* z dx \wedge dy &= \frac{1}{2\sqrt{|w|^2+1/4}^3}\left(1-\frac{|w|^2}{|w|^2+1/4}\right)dw^1 \wedge dw^2 \\
&= \frac{dw^1 \wedge dw^2}{8\sqrt{|w|^2+1/4}^5} \\
\sigma^* y dx \wedge dz &= -\frac{(w^2)^2}{2\sqrt{|w|^2+1/4}^5} dw^1 \wedge dw^2 \\
\sigma^* x dy \wedge dz &= \frac{(w^1)^2}{2\sqrt{|w|^2+1/4}^5} dw^1 \wedge dw^2.
\end{aligned}$$

We can add all three together to yield

$$\begin{aligned}
\sigma^*\omega &= \frac{|w|^2+1/4}{2\sqrt{|w|^2+1/4}^5} dw^1 \wedge dw^2 \\
&= \frac{dw^1 \wedge dw^2}{2\sqrt{|w|^2+1/4}^3}.
\end{aligned}$$

Now we can integrate in polar coordinates to yield

$$\int_{[B',C',D']} \sigma^*\omega = \int_0^{2\pi}\int_0^{\rho(\theta)} \frac{r}{2\sqrt{r^2+1/4}^3}\, dr\, d\theta.$$

Where ρ is the distance to the edge of the triangle in the direction θ. It is sufficient to note that the integrand is positive, and therefore the integral is positive. Lastly we note that ω is invariant under rotations, and $[B, C, D]$ can be transformed to every other face by an orientation preserving rotation. Hence the integrals around the other (oriented) faces are also positive, so the whole integral is positive.

□

3.

solution. We recall from lectures that f is closed, i.e. $df = 0$ implies f is constant. But if f is constant and compactly supported, then it is zero somewhere, and hence zero everywhere. Hence $H^0_c(\mathbb{R}^n) = 0$

□

4.

solution. 1. Recall that d "commutes" with the pulback operation, that is $f^*d = df^*$. Hence $f^*(\omega + d\tau) = f^*\omega + d^*d\tau$, hence $f^*\omega \sim f^*(\omega + d\tau)$, so the map is well defined for cohomology classes. That $f^* \circ g^* = (g \circ f)^*$ follows from the same fact for differential forms, which follows from $f^* \circ g^*\omega = f^* \circ \omega \circ g_* = \omega \circ g_* \circ f_* = \omega \circ (g \circ f)_* = (g \circ f)^* \circ \omega$.

2. Consider

$$\begin{aligned}(\omega + d\tau) \wedge (\zeta + d\xi) &= \omega \wedge \zeta + d\tau \wedge \zeta + \omega \wedge d\xi + d\tau \wedge d\xi \\ &= \omega \wedge \zeta + d(\tau \wedge \zeta + (-1)^{\deg \omega}\omega \wedge \xi + \tau \wedge d\xi) + (-1)^k \tau \wedge d\zeta - d\omega \wedge \xi + (-1)^k \tau \wedge d^2\xi \\ &= \omega \wedge \zeta + d(\tau \wedge \zeta + (-1)^{\deg \omega}\omega \wedge \xi + \tau \wedge d\xi).\end{aligned}$$

Hence the product class is invariant under the choice of representative factors.

□

5.

solution. By definition

$$\begin{aligned}\int_{f_*\sigma} \omega &= \int_{\Delta_k^\circ} (f_*\sigma \circ \rho_k)^*\omega \\ &= \int_{\Delta_k^\circ} (f \circ \sigma \circ \rho_k)^*\omega \\ &= \int_{\Delta_k^\circ} (\sigma \circ \rho_k)^* f^*\omega \\ &= \int_\sigma f^*\omega.\end{aligned}$$

□