Introduction to differential forms

model solutions 4

Jan Cristina

1.

solution. 1. By splitting our form into a sum of compactly supported prime forms, it suffices to show the claim for a form

$$\omega = f dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n.$$

Furthermore without loss of generality we may take i to be n, so that we prove the claim for $\omega = f dx^1 \wedge \cdots \wedge dx^{n-1}$, for any compactly supported C^1 function f. In this case

$$d\omega = \frac{\partial f}{\partial x^n} dx^1 \wedge \cdots \wedge dx^n,$$

hence

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x^n} dx.$$

Because f is compactly supported, so is $\partial_{x^n} f$, hence both are contained in some n-interval $I = (a_1, b_1) \times \cdots \times (a_n, b_n)$ We can then apply Fubini's theorem to yield

$$\begin{aligned}
\int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^{n-1}} \int_{a_n}^{b_n} \frac{\partial f}{\partial x^n}|_{(x,x_n)}\, dx^n \ldots dx \\
&= \int_{\mathbb{R}^{n-1}} f(x, b_n) - f(x, a_n)\, dx \\
&= \int_{\mathbb{R}^{n-1}} 0\, dx \\
&= 0.
\end{aligned}$$

We recall that because f is compactly supported with suppor contained entirely in I, that $f(x, b_n) = f(x, a_n) = 0$, because $(x, b_n), (x, a_n) \notin I$.

2. This is a simple application of the fact that

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

But by virtue of $\alpha \wedge \beta$ having compact support, the integral of its exterior derivative of satisfies

$$\int_{\mathbb{R}^n} d(\alpha \wedge \beta) = 0.$$

But expanding the exterior derivative yields

$$\int_{\mathbb{R}^n} d\alpha \wedge \beta + (-1)^k \int_{\mathbb{R}^n} \alpha \wedge d\beta = 0,$$

which when rearranged gives the result

$$\int_{\mathbb{R}^n} d\alpha \wedge \beta = (-1)^{k+1} \alpha \wedge d\beta.$$

□

2.

solution. Fix a basis $w_1, \ldots, w_n$ of $\mathbb{R}^n$ such that $w_1, \ldots, w_k$ forms a basis of W and $w_{k+1} = v$. Let α^i denote the dual basis. We may express a k-form ω as

$$\omega = \omega_W + \alpha_{k+1} \wedge \omega_v + \omega_0,$$

where

$$\omega_W = \omega_{[k]} \alpha^{[k]} \quad \omega_v = \sum_{\substack{J \subset k] \\ |J| = k-1}} \omega_{J \cup \{k+1\}} \alpha^J \qquad \omega_0 = \sum_{I \subset [n] // I \cap [n] \setminus [k+1] \neq \varnothing} \omega_I \alpha^I.$$

Then for any vector in for i the inclusion map $i : D \to \mathbb{R}^n$, $i^* \omega_0 = 0$. Simlarly $\alpha^{k+1} \wedge \omega_v | P = \alpha^{k+1} \wedge \omega_v | (P + v) = 0$. Hence we are need to show that

$$\int_{P+v} \omega_W - \int_P \omega_W = \int_D d(\omega_W + \alpha^{k+1} \wedge \omega_v).$$

In fact first we will show that

$$\int_D d(\alpha^{k+1} \wedge \omega_v) = 0.$$

To see this we note that

$$\begin{aligned}\int_D d(\alpha^{k+1} \wedge \omega_v) &= -\int_D \alpha^{k+1} \wedge d\omega_v \\ &= \int_D \alpha^{k+1} \wedge \sum_{i=1}^{k} \alpha^k \partial_k \omega_v. \\ &= \int_0^1 \int_{\mathbb{R}^k} dx^{k+1} \sum_{i=1}^{k} dx^k \wedge \Phi^* \partial_k \omega_v \\ &= \int_0^1 \int_{\mathbb{R}^k} d\Phi^* \omega_v \\ &= 0\end{aligned}$$

by the first problem, as ω was compactly supported and hence so is $\Phi^*\omega$. Now we must show that

$$\int_{P+v} \omega_W - \int_P \omega_W = \int_D d\omega_W.$$

For this we note that

$$\begin{aligned}\int_D d\omega_W &= \int_P \int_0^1 d\omega_W(v \wedge \xi)|_{x+tv}\, dt\, dx \\ &= \int_P \int_0^1 \partial_t \omega_{x+tv}(\xi)\, dt\, dx \\ &= \int_P \omega_{x+v}(\xi) - \omega_x(\xi)\, dx \\ &= \int_{P+v} \omega - \int_P \omega.\end{aligned}$$

□

3.

solution. 1. In order to clarify this we must explicitly state how the operators act. In this case $\theta_1 : e_i \mapsto \varepsilon^i$, and $\theta_2 : e_i \wedge e_j \mapsto \varepsilon^i \wedge \varepsilon^j$. The Hodge star acts in the following way:

$$\begin{cases} \star(\varepsilon^1) & = \varepsilon^2 \wedge \varepsilon^3 \\ \star(\varepsilon^2) & = \varepsilon^3 \wedge \varepsilon^1 \\ \star(\varepsilon^3) & = \varepsilon^1 \wedge \varepsilon^2 \end{cases}$$

and

$$\begin{cases} \star(\varepsilon^2 \wedge \varepsilon^3) & = \varepsilon^1 \\ \star(\varepsilon^3 \wedge \varepsilon^1) & = \varepsilon^2 \\ \star(\varepsilon^1 \wedge \varepsilon^2) & = \varepsilon^3. \end{cases}$$

We can write this more succinctly. Let (ijk) denote a permutation of (123) then we can express the star operator as

$$\star(\varepsilon^i) = \text{sgn }(ijk)\varepsilon^j \wedge \varepsilon^k \qquad \star(\varepsilon^i \wedge \varepsilon^j) = \text{sgn }(ijk)\varepsilon^k.$$

whereas for the cross product we have

$$e_i \times e_j = \text{sgn }(ijk)e_k$$

Now we wish to check

$$\begin{aligned} \star'(e_i \wedge e_j) &= \theta_1^{-1} \star (\varepsilon^i \wedge \varepsilon^j) \\ &= \theta_1^{-1}\text{sgn }(ijk)(\varepsilon^k) \\ &= \text{sgn }(ijk)e_k. \end{aligned}$$

Hence the two are equal.

2. The curl-operator is given by

$$\text{curl} \sum_i X^i(x)e_i = \sum_{\sigma \in S^3} \text{sgn }(\sigma)e_{\sigma(1)}\partial_{x^{\sigma(2)}}X^{\sigma(3)}(x).$$

In this case curl can be expressed as $\theta_1^{-1} \star d\theta_1$.

$$\begin{aligned} \theta_1^{-1} \star d\theta_1(\sum_i X^i e_i) &= \theta_1^{-1} \star d(\sum_i X^i \varepsilon^i) \\ &= \theta_1^{-1} \star \sum_{i,j} \partial_j X^i \varepsilon^j \wedge \varepsilon^i \\ &= \theta_1^{-1} \sum_{i,j} \partial_j X^i \varepsilon^k \text{sgn }(jik) \\ &= \sum_{i,j} \text{sgn }(jik)\partial_j X^i e_k \\ &= \sum_{\sigma \in S^3} \text{sgn }(\sigma)e_{\sigma(1)}\partial_{\sigma(2)}X^{\sigma(3)}. \end{aligned}$$

3. The divergence operator for a vector field $X = \sum_i X^i(x)e_i$ is given be

$$\text{div } X = \sum_i \partial_i X^i.$$

Let us calculate

$$\begin{aligned}
\star d(X \llcorner dx^1 \wedge \cdots \wedge dx^n) &= \star d(\sum_i X^i e_i \llcorner (-1)^{i-1} dx^i \wedge dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n \\
&= \star d \sum_i (-1)^{i-1} X^i dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n \\
&= \star \sum_i (-1)^{i-1} \partial_i X^i dx^i \wedge dx^1 \wedge \cdots \wedge d\hat{x}^i \wedge \cdots \wedge dx^n \\
&= \star \sum_i \partial_i X_i dx^1 \wedge \cdots dx^n \\
&= \sum_i \partial_i X_i.
\end{aligned}$$

□

4.

solution. Suppose ω is a closed compactly supported k-form for $k < n$. Then by problem 2

$$\int_{\mathbb{R}^k \times \{0\}} \omega - \int_{\mathbb{R}^k \times \{v\}} \omega = \int_D d\omega = \int_D 0 = 0$$

where $D = \{(x, tv) : x \in \mathbb{R}^k \, t \in [0, 1]\}$. Hence

$$\int_{\mathbb{R}^k} \times \{0\} \omega = \int_{\mathbb{R}^k \times \{v\}} \omega.$$

But for $|v|$ sufficiently large, $\mathbb{R}^k \times \{v\}$ does not intersect the support of ω, hence $\int_{\mathbb{R}^k \times \{v\}} \omega = 0$, but then $\int_{\mathbb{R}^k \times \{0\}} \omega = 0$.

Hence $\xi = 0$ suffices. For $k = n$, the trivial 0 form 1 suffices. □

5.

solution. 1. To show this we note that

$$\begin{aligned}
d^* d^* &= (-1)^{nk-1} \star d \star (-1)^{n(k+1)-1} \star d \star \\
&= (-1)^{2nk+n-2} \star d \star^2 d \star \\
&= (-1)^n (-1)^{k(n-k)} \star d^2 \star \\
&= 0.
\end{aligned}$$

2. We once again calculate

$$\begin{aligned}
d^*du &= -1 \star d \star \sum_i \partial_i u dx^i \\
&= - \star d \sum_i \partial_i (-1)^{i-1} dx^{[n]\setminus\{i\}} \\
&= - \star \sum_i (-1)^{i-1} \partial_i (\partial_i u) dx^i \wedge dx^{[n]\setminus\{i\}} \\
&= - \star \sum_i \partial_i^2 u dx^{[n]} \\
&= \sum_i D_i^2 u \\
&= -\Delta u.
\end{aligned}$$

3. It suffices to check this for a prime k-form $\alpha = udx^I$. We define $\sigma(I, J)$ by $\star dx^I \wedge dx^J =$

$\sigma(I)dx^{I\cup J}$, and $\sigma(I) := \sigma(I,[n]\setminus I)$.

$$\begin{aligned}
(d^*d + dd^*)udx^I &= ((-1)^{nk-1}\star d \star d + d(-1)^{n(k+1)-1}\star d\star)u\,dx^I \\
&= (-1)^{nk-1}\star d\star \sum_{i\in[n]\setminus I}\sigma(i,I)\partial_i u\,dx^{I\cup i} \\
&\quad + (-1)^{n(k+1)-1} d\star d\sigma(I)u\,dx^{[n]\setminus I} \\
&= (-1)^{nk-1}\star d \sum_{i\in[n]\setminus I}\sigma(i,I)\sigma(I\cup i)\partial_i u\,dx^{[n]\setminus(I\cup i)} \\
&\quad + (-1)^{n(k+1)-1} d\star \sum_{i\in I}\sigma(I)\sigma(i,[n]\setminus I)\partial_i u\,dx^{([n]\setminus I)\cup i} \\
&= (-1)^{nk-1}\star \sum_{i\in[n]\setminus I}\sum_{j\in I\cup i}\sigma(i,I)\sigma(I\cup i)\sigma(j,[n]\setminus(I\cup i))\partial_j\partial_i u\,dx^{[n]\setminus(I\cup i)\cup j} \\
&\quad + (-1)^{n(k+1)} d\sum_{i\in I}\sigma(I)\sigma(i,[n]\setminus I)\sigma(([n]\setminus I)\cup i)\partial_i u\,dx^{I\setminus i} \\
&= (-1)^{nk-1}\sum_{i\in[n]\setminus I}\sum_{j\in I\cup i}\sigma(i,I)\sigma(I\cup i)\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)\partial_j\partial_i u\,dx^{(I\cup i)\setminus j} \\
&\quad + (-1)^{n(k+1)-1}\sum_{i\in I}\sum_{j\in[n]\setminus I\cup i}\sigma(I)\sigma(i,[n]\setminus I)\sigma(([n]\setminus I)\cup i)\sigma(j,I\setminus i)\partial_j\partial_i u\,dx^{I\setminus i\cup j} \\
&= (-1)^{nk-1}\sum_{i\in[n]\setminus I}\sigma(i,I)\sigma(I\cup i)\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I)\partial_i^2 u\,dx^I \\
&\quad + (-1)^{n(k+1)-1}\sum_{i\in I}\sigma(I)\sigma(i,[n]\setminus I)\sigma(([n]\setminus I)\cup i)\sigma(i,I\setminus i)\partial_i^2 u\,dx^I \\
&\quad + (-1)^{nk-1}\sum_{i\in[n]\setminus I}\sum_{j\in I}\sigma(i,I)\sigma(I\cup i)\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)\partial_j\partial_i u\,dx^{I\setminus i\cup j} \\
&\quad + (-1)^{n(k+1)-1}\sum_{j\in I}\sum_{i\in[n]\setminus I}\sigma(I)\sigma(j,[n]\setminus I)\sigma(([n]\setminus I)\cup j)\sigma(i,I\setminus j)\partial_j\partial_i u\,dx^{I\setminus i\cup j}
\end{aligned}$$

The result will pop out if we can show the following

$$\begin{aligned}
-1 &= (-1)^{nk-1}\sigma(i,I)\sigma(I\cup i)\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I) \\
-1 &= (-1)^{n(k+1)-1}\sigma(I)\sigma(i,[n]\setminus I)\sigma([n]\setminus I\cup i)\sigma(i,I\setminus i) \\
0 &= (-1)^{nk-1}\sigma(i,I)\sigma(I\cup i)\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j) \\
&\quad + (-1)^{n(k+1)-1}\sigma(I)\sigma(j,[n]\setminus I)\sigma(([n]\setminus I)\cup j)\sigma(i,I\setminus j).
\end{aligned}$$

To calculate this we note that $\sigma(i,I)dx^i\wedge dx^I = dx^{I\cup I}$ and $\sigma(I\cup i)dx^{I\cup i}\wedge dx^{[n]\setminus(I\cup i)} = dx^{[n]}$. Combining these we get

$$\sigma(i,I)\sigma(I\cup i)dx^i\wedge dx^I\wedge dx^{[n]\setminus(I\cup i)} = dx^{[n]}$$

We can similarly calculate to get

$$\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I)dx^i\wedge dx^{[n]\setminus(I\cup i)}\wedge dx^I=dx^n.$$

We can equate these to to get

$$\begin{aligned}\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I)dx^i\wedge dx^{[n]\setminus(I\cup i)}\wedge dx^I&\\ =\sigma(i,I)\sigma(I\cup i)dx^i\wedge dx^I\wedge dx^{[n]\setminus(I\cup i)}&\end{aligned}\tag{1}$$

$$\begin{aligned}(-1)^{k(n-k-1)}\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I)dx^i\wedge dx^I\wedge dx^{[n]\setminus(I\cup i)}&\\ =\sigma(i,I)\sigma(I\cup i)dx^i\wedge dx^I\wedge dx^{[n]\setminus(I\cup i)}&\end{aligned}\tag{2}$$

Cancelling out we get that

$$\sigma(i,I)\sigma(I\cup i)\sigma(i,[n]\setminus(I\cup i))\sigma([n]\setminus I)=(-1)^{k(n-k-1)}$$

Then $(-1)^{k(n-k-1)}(-1)^{nk-1}=(-1)^{k^2-k-1}=(-1)^{k(k-1)-1}=(-1)^{-1}=-1$. Similarly we get that

$$\sigma(I)\sigma(i,[n]\setminus I)\sigma([n]\setminus I\cup i)\sigma(i,I\setminus i)=(-1)^{(k-1)(n-k)}$$

and so $(-1)^{n(k+1)-1}(-1)^{(k-1)(n-k)}=(-1)^{-k^2+k-1}=(-1)^{-k(k-1)-1}=-1$

For the other terms we note that once again

$$\begin{aligned}dx^{[n]}&=\sigma(i,I)\sigma(I\cup i)dx^i\wedge dx^I\wedge dx^{[n]\setminus(I\cup i)}\\&=\sigma(i,I)\sigma(I\cup i)dx^i\wedge\sigma(j,I\setminus j)dx^j\wedge dx^{I\setminus j}\wedge dx^{[n]\setminus(I\cup i)}\end{aligned}$$

and

$$\begin{aligned}dx^{[n]}&=\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)dx^j\wedge dx^{[n]\setminus I\cup i}dx^{(I\cup i)\setminus j}\\&=\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)dx^j\wedge dx^{[n]\setminus I\cup i}\sigma(i,I\setminus j)dx^i\wedge dx^{I\setminus j}.\end{aligned}$$

Rearranging yields

$$\begin{aligned}\sigma(i,I)\sigma(I\cup i)\sigma(j,I\setminus j)dx^i\wedge dx^j\wedge dx^{I\setminus j}\wedge dx^{[n]\setminus(I\cup i)}&\\ =(-1)^{n-k+(k-1)(n-k-1)}\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)\sigma(i,I\setminus j).&\end{aligned}\tag{3}$$

from which we get that

$$\begin{aligned}\sigma(i,I)\sigma(I\cup i)\sigma(j,[n]\setminus(I\cup i))\sigma([n]\setminus(I\cup i)\cup j)&\\ =(-1)^{nk-k^2-k+1}\sigma(j,I\setminus j)\sigma(i,I\setminus j).&\end{aligned}\tag{4}$$

Now let us consider the second term

$$\begin{aligned} dx^{[n]} &= \sigma(I)\sigma(j, I\setminus j)dx^j \wedge dx^{I\setminus j} \wedge dx^{[n]\setminus I} \\ &= (-1)^{(k-1)(n-k)}\sigma(I)\sigma(j, I\setminus j)dx^j \wedge dx^{[n]\setminus I} \wedge dx^{I\setminus j} \\ &= (-1)^{(k-1)(n-k)}\sigma(I)\sigma(j, I\setminus j)dx^j \wedge dx^{[n]\setminus I} \wedge dx^{I\setminus j} \\ &= (-1)^{(k-1)(n-k)}\sigma(I)\sigma(j, I\setminus j)\sigma(j,[n]\setminus I)\sigma(([n]\setminus I)\cup j)dx^n. \end{aligned}$$

Hence we have that

$$\sigma(I)\sigma(j,[n]\setminus I)\sigma(([n]\setminus I)\sigma(i, I\setminus j) = (-1)^{(k-1)(n-k)}\sigma(j, I\setminus j)\sigma(i, I\setminus j).$$

Now we only need to examine

$$\begin{aligned} (-1)^{(k-1)(n-k)}(-1)^{n(k+1)-1} + (-1)^{nk-1}(-1)^{nk-k^2-k+1} &= (-1)^{(k+1)(-k)-1} + (-1)^{-k^2-k} \\ &= (-1)^{-k^2-k-1} + (-1)^{-k^2-k} \\ &= 0. \end{aligned}$$

This proves the claim.

□

6.

solution. 1. Define the map $\partial[0,1]^3 \to S^2$ by

$$x \mapsto \frac{x - (1/2, 1/2, 1/2)}{|x - (1/2, 1/2, 1/2)|}.$$

This is a map from the boundary of the unit cube to the unite sphere. It is continuous from a compact space, and bijective, and hence a homeomorphism. To see that it is a bijection one need only note that $[0,1]^3$ is a convex set, so a ray from $(1/2, 1/2, 1/2)$ intersects the boundary at exactly one point. The ray then intersects the unit sphere centered at $(1/2, 1/2, 1/2)$ at exactly one point. This is the bijective correspondence.

Thus we only need to find a manifold structure for S^2. For this we take the stere-graphic projections $\phi_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ and $\phi_S : S^2 \setminus \{S\} \to \mathbb{R}^2$, where $N = (0,0,1)$ and $S = (0,0,-1)$.

Using polar coordinates for $\mathbb{R}^2$ we have

$$\phi_N^{-1}(r,\phi) = \frac{1}{r^2+1}\begin{pmatrix} 2r\cos\phi \\ 2r\sin\phi \\ r^2-1 \end{pmatrix}.$$

While ϕ_S takes the following form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto (\sqrt{1-z^2}/(1+z), \arctan(y/x))$$

under this map the composition takes $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}^2 \setminus \{0\}$ $(r, \phi) \mapsto (1/r, \phi)$.

2. We note that the map $\mathbb{R}^2 \to T$

$$(s, t) \mapsto (\cos(s)(R + r\cos(t)), \sin(s)(R + r\cos(t)), r\sin(t))$$

fibres through $\mathbb{R} \times S^1$ via the map

$$\phi : (t, \tau) \mapsto (\Re(\tau)(R + r\cos(t)), \Im(\tau)(R + r\cos(t)), r\sin(t)).$$

Then the sets

$$U_1 = \{(t, \tau) : 0 < t < 2\pi\}$$
$$U_2 = \{(t, \tau) : \pi < t < 3\pi\},$$

are homeomorphic to complex (and hence Euclidea) domains via the map $(t, \tau) \mapsto t\tau$. They map injectively onto open sets of T. To see this we note only that τ is uniquely determined by the x and y coordinates. And t is determined modulo 2π.

As such $\phi|_{U_1}^{-1}(\phi(U_1) \cap \phi(U_2)) = U_1 \setminus \{\pi\} \times \tau$, and $\phi|_{U_2}^{-1} \circ \phi(t, \tau) = (t, \tau)$ if $\pi < \tau < 2\pi$ and $\phi|_{U_2}^{-1} \circ \phi(t, \tau) = (t + 2\pi, \tau)$ if $0 < t < \pi$. Hence this is a smooth manifold.

□