Introduction to differential forms

model solutions 2

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1.

solution. 1. In general when faced with a proof for an insurmountable number of indices, of a general length, it is good to proceed with induction. In this case we can make the following inductive hypothesis for all $(n \times n)$-matrices, B we have that

$$\det B = \sum_{\sigma \in S_n} \text{sgn}\,(\sigma) b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

Now assume for all $k \in \mathbb{N}$ and for all $(k \times k)$-matrices A

$$\det A = \sum_{i=1}^{k} (-1)^i a_{1i} \det A_{1i}.$$

Now let $k = n + 1$ then

$$\det A = \sum_{i=1}^{n+1} (-1)^i a_{1i} \sum_{\sigma \in S_{n+1}, \sigma(1)=1} \text{sgn}\,(\sigma) a_{2(1,i)\sigma(2)} \cdots a_{n+1(1,i)\sigma(n+1)}$$

$$\sum_{\sigma \in S_{n+1}} \text{sgn}\,(\sigma) a_{1\sigma(1) \cdots a_{n\sigma(n)}}.$$

Then because the determinant formula is trivial for $n = 1$, by induction we have the result.

2. We have by definition that

$$\begin{aligned} \varepsilon_1 \wedge \cdots \wedge \varepsilon_n(v_1, \ldots, v_n) &= \sum_{\sigma \in S_n} \text{sgn}\,(\sigma) \varepsilon_1(v_{\sigma(1)}) \cdots \varepsilon_n(v_{\sigma(n)}) \\ &= \sum_{\sigma \in S_n} \text{sgn}\,(\sigma) v_{1\sigma(1)} \cdots v_{n\sigma(n)} \\ &= \det(v_1, \ldots, v_n). \end{aligned}$$

3. Now we must show that any alternating k-linear map α can be represented as

$$\sum_I \alpha_I f_I^* \det,$$

where det is the standard determinant in $\mathbb{R}^k$. To do this, we define

$$\alpha_I = \alpha(e_{i_1}, \ldots, e_{i_k}).$$

Then we can examine

$$\begin{aligned}(\alpha)(v_1, \ldots, v_k) &= \sum_{i_1,\ldots,i_k} v_{1i_1} \cdots v_{ki_k} \alpha(e_{i_1}, \ldots, e_{i_k}) \\ &= \sum_{I \subset N} \sum_{\sigma \in S_I} \text{sgn}\,(\sigma) v_{1\sigma(i_1)} \cdots v_{k\sigma(i_k)} \alpha_I \\ &= \sum_{I \subset N} \sum_{\sigma \in S_k} \text{sgn}\,(\sigma) f_I(v_1)_{\sigma(1)} \cdots f_I(v_k)_{\sigma(k)} \alpha_I \\ &= \sum_{I \subset N} \alpha_I \det(f_I(v_1), \ldots, f_I(v_k)) \\ &= \sum_{I \subset N} \alpha_I f_I^* \det(v_1, \ldots, v_k).\end{aligned}$$

□

2.

solution. □

1. Suppose $v_k = \sum_{i=1}^{k-1} \lambda_i v_i$. Then

$$\begin{aligned}\omega(v_1, \ldots, v_k) &= \sum_{i=1}^{k-1} \lambda_i \omega(v_1, \ldots, v_{k-1}, v_i) \\ &= 0.\end{aligned}$$

as the two of the entries are equal. As a conclusion any alternating k-linear map on V is 0 for $\dim V < k$ as any collection of k vectors is linearly dependent.

2. We denote the permutation on $k + l$ symbols which maps i to $l + i$ for $1 \leq i \leq k$, and maps $k + j$ to j for $1 \leq j \leq l$, by ς. This permutation is idempotent, that is $\varsigma^2 = id$,

and has sign sgn $(\varsigma) = (-1)^{lk}$. We continue with the definition of the wedge product

$$\begin{aligned}
\omega \wedge \tau(v_1, \ldots, v_{k+l}) &= \sum_{\sigma \in S_{k+l,k}} \text{sgn }(\sigma\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)} \ldots, v_{\sigma(k+l)}) \\
&= \sum_{\sigma \in S_{k+l,k}} \text{sgn }(\sigma)\tau(v_{\sigma\circ\varsigma(1)}, \ldots, v_{\sigma\circ\varsigma(l)}\omega(v_{\sigma\circ\varsigma(l+1)}, \ldots, v_{\sigma\circ\varsigma(l+k)} \\
&= \sum_{\sigma \in S_{k+l,k}} (-1)^{lk}\text{sgn }(\sigma \circ \varsigma)\tau(v_{\sigma\circ\varsigma(1)}, \ldots, v_{\sigma\circ\varsigma(l)}\omega(v_{\sigma\circ\varsigma(l+1)}, \ldots, v_{\sigma\circ\varsigma(l+k)}) \\
&= (-1)^{lk} \sum_{\rho \in S_{k+l,l}} \text{sgn }(\rho)\tau(v_{\rho(1)}, \ldots, v_{\rho(l)})\omega(v_{\rho(l+1)}, \ldots, v_{\rho(k+l)}) \\
&= \tau \wedge \omega(v_1, \ldots, v_{k+l}).
\end{aligned}$$

3.

solution. 1. Suppose $n = 2$, then there is precisely one subset of the set of two elements, with two elements, namely, the set itself. This is important for applying the structure theorem. The condition that $L^*(\alpha) = \alpha$ in $\mathbb{R}^2$ says precisely that $L^*\alpha(v_1, v_1) = \alpha(v_1, v_2)$, which by the structure theorem gives $\alpha(v_1, v_2) = \alpha \det(v_1, v_2)$. But $L^*\alpha(v_1, v_2) = \alpha(L(v_1), L(v_2)) = \alpha \det L \det(v_1, v_2)$. Thus $\det L = 1$ is equivalent to L^* is the identity on Alt^k.

2. Let ε_i $i = 1, 2, 3$ be a basis of $\mathbb{R}^{3*}$. Consider $A^*(\varepsilon_1) \wedge A^*(\varepsilon_2 \wedge \varepsilon_3) = A^*(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3) = \det(A)\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$. With this we can see that $A^*\varepsilon_1 = \varepsilon_1 + a_{21}\varepsilon_2 + b_{31}\varepsilon_3$. Similarly for $A^*\varepsilon_2$ and $A^*\varepsilon_3$. Now consider

$$\begin{aligned}
\varepsilon_1 \wedge \varepsilon_2 &= A^*\varepsilon_1 \wedge A^*\varepsilon_2 \\
&= (\varepsilon_1 + a_{21}\varepsilon_2 + a_{31}\varepsilon_3) \wedge (a_{12}\varepsilon_1 + \varepsilon_2 + a_{32}\varepsilon_3) \\
&= (1 - a_{12}a_{21})\varepsilon_1 \wedge \varepsilon_2 + (a_{32} - a_{31}a_{12})\varepsilon_1 \wedge \varepsilon_3 + (a_{21}a_{32} - a_{31})\varepsilon_2 \wedge \varepsilon_3.
\end{aligned}$$

From this we can conclude that

$$a_{12}a_{21} = 0 \quad a_{32} = a_{31}a_{12} \quad a_{31} = a_{32}a_{21}.$$

But by examining $\varepsilon_2 \wedge \varepsilon_3$ and $\varepsilon_3 \wedge \varepsilon_1$ we can arrive at

$$a_{23}a_{32} = 0 \quad a_{12} = a_{13}a_{32} \quad a_{13} = a_{12}a_{23},$$

$$a_{31}a_{13} = 0 \quad a_{21} = a_{23}a_{31} \quad a_{23} = a_{21}a_{13}.$$

Suppose $a_{12} = 0$. Then $a_{32} = 0$, and $a_{13} = 0$. Then $a_{31} = 0$, and $a_{23} = 0$. Then $a_{21} = 0$ and $a_{13} = 0$. Hence $A = I$.

3. This problem is no longer assigned

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4.

solution. 1. Assuming $\Psi = \Phi$, and denoting by S_I the set of injective maps from the set of k symbols to $\{i_1 < \ldots < i_k\} \subset N$ with sgn $: S_I \to \{\pm 1\}$ given by sgn $(f \circ \sigma) =$ sgn (σ) where f is the map $j \mapsto i_j$, and $\sigma \in S_k$ we have that

$$\langle \Psi(v_1) \wedge \cdots \wedge \Psi(v_k), \Psi(w_1) \wedge \cdots \wedge \Psi(w_k) \rangle = \left\langle \sum_{i_1,\ldots,i_k} v_{1i_j}\varepsilon_{i_1} \wedge \cdots \wedge v_{ki_k}\varepsilon_{i_k}, \sum_{j_1,\ldots,j_k} w_{1j_j}\varepsilon_{j_1} \wedge \cdots \wedge w_{kj_k}\varepsilon_{j_k} \right\rangle$$

$$\begin{aligned}
&= \left\langle \sum_{i_1,\ldots,i_k} v_{1i_1}\cdots v_{ki_k}\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}, \sum_{j_1,\ldots,j_k} w_{1j_1}\cdots w_{kj_k}\varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k} \right\rangle \\
&= \sum_{i_1,\ldots i_k} \sum_{\sigma S_k} \text{sgn}\,(\sigma) v_{1i_1}\cdots v_{ki_k} w_{1i_{\sigma(1)}}\cdots w_{ki_{\sigma(k)}}. \\
&= \sum_{\sigma \in S_k} \text{sgn}\,(\sigma) \sum_{i_1,\ldots,i_k} v_{1i_1}\cdots v_{ki_k} w_{\sigma(1)i_1}\cdots w_{\sigma(k)i_k} \\
&= \det\langle v_i, w_j \rangle
\end{aligned}$$

2. The appropriate map needs to take e_I to some mutiple of $e_{N\setminus I}$, where $N = \{1, \ldots, n\}$. Then we must find the appropriate scaling. We will call it sgn (n, I) and is given by sgn (σ) for $\sigma \in S_{n,k}$, where $\sigma(j) = i_j$ for $1 \le j \le k$. Let us test this

$$e_I \wedge *e_J = \text{sgn}\,(\sigma) e_I \wedge e_J = \begin{cases} 0 & \text{if } I \neq J \\ \text{sgn}\,(\sigma) e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{i_{k+1}} \wedge \cdots e_{i_n}. \end{cases}$$

Now σ puts $i_1, \ldots, i_n$ in the right order, hence this has the desired property. The property is linear, and hence holds for every vector (fix ω). Now we could suppose that $\omega \wedge B\eta = \langle \omega, \eta \rangle \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$, but by subtracting B and $*$ we may as well suppose that $\omega \wedge A\eta = 0$ for every ω and η. Now fix $\zeta = A\eta$, and suppose $\omega \wedge \zeta = 0$ for every ω, then we wish to show that $\zeta = 0$. But $\varepsilon_I \wedge \zeta = \zeta_{N\setminus I}\varepsilon_I \wedge \varepsilon_{N\setminus I}$. If this is 0 for every I, then $\zeta = 0$.

□

5.

solution. 1. Now it is sufficient to examine the action on a basis element. Consider $*(e^I) = \text{sgn}\,(n, I)e^{N\setminus I}$, then $*(e^{N\setminus I}) = \text{sgn}\,(n, N \setminus I)e^{N\setminus(N\setminus I)} = \text{sgn}\,(n, N \setminus I)e^I$. Hence $** (e^I) = \text{sgn}\,(n, I)\text{sgn}\,(n, N \setminus I)e^I$. Now let us consider $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_n$, and let $\sigma(j) = i_j$, then $\text{sgn}\,(n, I) = \text{sgn}\,(n, I)$ whereas $\varsigma(j) = i_{k+j}$ for $1 \leq j \leq n-k$, and $\varsigma(j) = i_{j-(n-k)}$ for $n-k < j < n$, and $\text{sgn}\,(\varsigma) = \text{sgn}\,(n, N \setminus I)$. Then $\text{sgn}\,(n, I)\text{sgn}\,(n, N \setminus I) = \text{sgn}\,(\varsigma^{-1} \circ \sigma)$, and the maps $\varsigma^{-1} \circ \sigma$ takes j to $j + (n-k)$ for $1 \leq j \leq k$ and to $j - k$ for $k < j \leq n$, and has sign $(-1)^{k(n-k)}$.

2. From now on denote $\varepsilon_N = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$, whenever n is the dimension of the space. This should read $A^* \circ * = * \circ A^* : \text{Alt}^k(\mathbb{R}^n) \to \text{Alt}^{n-k}(\mathbb{R}^m)$. Now consider

$$\begin{aligned}
\omega \wedge A^*(*\eta)(v_1, \ldots, v_n) &= \overbrace{\det A^t}^{=1}\, \omega \wedge A^*(*\eta)(v_1, \ldots, v_n) \\
&= \omega \wedge A^*(*\eta)(A^t v_1, \ldots, A^t v_n) \\
&= A^{t*}\omega \wedge A^{t*}A^*(*\eta)(v_1, \ldots, A^t v_n) \\
&= A^{t*}\omega \wedge *\eta(v_1, \ldots, v_n). \\
&= \langle A^{t*}\omega, \eta\rangle(v_1, \ldots, v_n).
\end{aligned}$$

Now we must show that $\langle L^{t*}\omega, \eta\rangle = \langle \omega, L^*\eta\rangle$ for any linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$. To see this consider

$$\begin{aligned}
\langle L^*\varepsilon_I, \varepsilon_J\rangle &= \varepsilon_I(Le_{j_1}, \ldots, Le_{j_k}) \\
&= \sum_{\sigma \in S_k} \text{sgn}\,(\sigma)\varepsilon_{i_1}(Le_{j_{\sigma(1)}}) \cdots \varepsilon_{i_k}(Le_{j_{\sigma(k)}}) \\
&= \sum_{\sigma \in S_k} \text{sgn}\,(\sigma)\langle e_{i_1}, Le_{j_{\sigma(1)}}\rangle \cdots \langle e_{i_k}, Le_{j_{\sigma(k)}}\rangle \\
&= \sum_{\sigma \in S_k} \text{sgn}\,(\sigma)\langle e_{i_{\sigma(1)}}, Le_{j_1}\rangle \cdots \langle L^t e_{i_{\sigma(k)}}, e_{j_k}\rangle \\
&= \langle \varepsilon_I, L^{t*}\varepsilon_J\rangle.
\end{aligned}$$

Now we merely note that $\langle A^{t*}\omega, \eta\rangle = \langle \omega, A^*\eta\rangle$ to arrive at

$$\omega \wedge A^*(*\eta) = \omega \wedge *(A^*\eta).$$

□

6.

solution. 1. We note that a Euclidean ball is a convex set so for any $x, y \in B$ ant $s \in [0, 1]$ we have that $sx + (t-s)y \in B$. Using this we draw a straight line between $\gamma(t)$ and $\tilde{\gamma}(t)$ given by

$$H(s, t) = s\gamma(t) + (1-s)[(1-t)\gamma(0) + t\gamma(1)].$$

By convexity it is always in B, and for $t = 0, 1$ the path in s is constant.

2. Because any curve $\gamma : \mathbb{R}^3 \setminus \{0\}$ has a compact image, and so has positive distance from 0. Furthermore it is uniformly continuous, hence there is a piecewise linear approximation of γ, which we call γ_ε for which $\|\gamma \setminus \gamma_\varepsilon\|_\infty < \varepsilon$ Then the straight line homotopy $s\gamma_\varepsilon(t) + (1-s)\gamma(t)$ misses 0. Now if we look at this curve projected to the unit sphere, it misses a point, θ_0, because it has finite length. Now in spherical coordinates $\gamma = (\rho, \theta)$. Then we take the homotopy given by $(s,t) \mapsto ((1-s)\rho(t) + s\rho(0), \phi_s(\theta(t)))$, where $\phi(s)$ is the Möbius map with sink $\theta(0)$, and source θ_0. (Under stereographic projection taking θ_0 to ∞, they are given by

$$\phi_s = (1-s)\theta(t) + s\theta(0).$$

$\square$