Introduction to differential forms

model solutions

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1.

solution. This problem would be easier if the condition

$$\int_0^1 \langle X(\gamma(t)), \dot{\gamma}(t)\rangle \, dt = \int_0^1 \langle Y(\gamma(t), \dot{\gamma(t)}\rangle \, dt \tag{1}$$

were stated to hold for every *piecewise* C^1-loop. But we need not worry as in fact the two are equivalent. We can rewrite (1) as

$$\int_0^1 \langle Z(\gamma(t)), \dot{\gamma}(t)\rangle \, dt = 0,$$

where $Z = X - Y$. There are at least two ways of doing this. One is the method of smooth mollifiers, whereby we can construct smooth approximations to piecewise C^1 curves, and another is to explicitly construct a C^1 smooth curve which is equivalent to a piecewise C^1. The way to do this is to construct a new parametrisation whose derivative vanishes at the points of discontinuity. To do this we will need a change of variables $(0,1) \to (0,1)$ such that the derivative vanishes at 0 and 1. We may as well see if a polynomial fits our needs, so we need a polynomial $p(x)$ for which $p(0) = 0$, $p(1) = 1$, $p'(0) = 0$, $p'(1) = 0$. That is four equations, so we need a minimum of four parameters, so a cubic equation

$$p(x) = a\,x^3 + b\,x^2 + c\,x + d,$$

yielding

$$\begin{array}{ccccccccc} & & & & & & d & = & 0 \\ a & + & b & + & c & + & d & = & 1 \\ & & & & c & & & = & 0 \\ 3\,a & + & 2\,b & + & c & & & = & 0 \end{array}.$$

Solving this yields $a = -2$ and$b = 3$, hence $p(x) = -2x^3 + 3x^2$. Furthermore it is easy to verify that $p'(x) > 0$ for $0 < x < 1$, so $p(x) : [0,1] \to [0,1]$ homeomorphicall, and diffeomorphically on the interior. Now let $\gamma : [0,1] \to \mathbb{R}^n$ be a piecewise C^1 curve with derivative discontinuous on $0 < t_1 < \cdots < t_k < 1$. Now consider the the curve

$$\sigma_{k-1} : t \mapsto \begin{cases} (\gamma(t_k p(t/t_k)) & 0 \le t < t_k \\ (\gamma((1-t_k)p[(t-t_k)/(1-t_k)] + t_k) & t_k \le t < 1. \end{cases}$$

Now the curve sigma is clearly C^1 for $t_k p(t/t_k) \neq t_i$ where $i = 1, \ldots k-1$. which by the diffeomorphism property of p in the interior holds for precisely $k-1$ points. Furthermore σ_{k-1} is C^1 for $t_k < t < 1$. Lastly we must show that σ_{k-1} has continuous derivative at t_k. But it's derivative is given by

$$\sigma'_{k-1} : t \mapsto \begin{cases} (\gamma'(t_k p(t/t_k))p'(t/t_k) & 0 \le t < t_k \\ (\gamma'((1-t_k)p[(t-t_k)/(1-t_k)] + t_k)p'[(t-t_k)/(1-t_k)] & t_k \le t < 1. \end{cases}$$

Now γ has finite left and right limits around t_k, and p' has value 0, so σ'_{k-1} is continuous at t_k and has limit 0.

Now relabel $p_1(t) := t_k p(t/t_k)$ and $p_2(t) := (1-t_k)p[(t-t_k)/(1-t_k)]+t_k$. If we calculate the integral condition for σ_{k-1} we arrive at

$$\begin{aligned} \int_0^1 \langle Z(\sigma_{k-1}(t), \sigma'_{k-1}(t)\rangle \, dt &= \int_0^{t_k} \langle Z(\sigma_{k-1}(t), \sigma'_{k-1}(t)\rangle \, dt + \int_{t_k}^1 \langle Z(\sigma_{k-1}(t)), \sigma'_{k-1}(t)\rangle \, dt. \\ &= \int_0^{t_k} \langle Z(\gamma \circ p_1(t)), \dot\gamma \circ p_1(t) p'_1(t)\rangle \, dt + \int_{t_k}^1 \langle Z(\gamma \circ p_2(t)), \dot\gamma \circ p_2(t) p'_2(t)\rangle \, dt, \\ &= \int_0^{t_k} \langle Z(\gamma \circ p_1(t)), \dot\gamma \circ p_1(t)\rangle p'_1(t) \, dt + \int_{t_k}^1 \langle Z(\gamma \circ p_2(t)), \dot\gamma \circ p_2(t)\rangle p'_2(t) \, dt, \end{aligned}$$

We can then apply the change of variable formula which yields

$$\begin{aligned} \int_0^1 \langle Z(\sigma_{k-1}(t), \sigma'_{k-1}(t)\rangle \, dt &= \int_0^{t_k} \langle Z(\gamma(t)), \dot\gamma(t)\rangle \, dt + \int_{t_k}^1 \langle Z(\gamma(t)), \dot\gamma(t)\rangle \, dt \\ &= \int_0^1 \langle Z(\gamma(t)), \dot\gamma(t)\rangle \, dt. \end{aligned}$$

This can be repeated inductively, so that we can construct a map σ_0 which is C_1 for which

$$\int_0^1 \langle Z(\sigma_0(t), \sigma'_0(t)\rangle \, dt = \int_0^1 \langle Z(\gamma(t)), \dot\gamma(t)\rangle \, dt.$$

Thus we know that property (1) holds for C^1 loops if and only if it holds for piecewise C^1 loops.

Now we want to show that if Z has property (1), for all piecewise C^1 loops then $Z = \nabla f$ for some f. Clearly the f wont be unique as we can always add a constant C, in which case $\nabla f = \nabla f + \nabla C$.

Let γ_1 and γ_2 be two loops with initial point $x_0 \in \Omega$ endpoint $x_1 \in \Omega$. Then denote $\bar{\gamma}_2$ the curve $\gamma_2(1-t)$. In this case

$$\int_0^1 \langle Z(\gamma_1(t)), \dot{\gamma}_1 \rangle \, dt = \int_0^1 \langle Z(\gamma_2(t)), \dot{\gamma}_2 \rangle \, dt.$$

This is because the curve σ given by following γ_1 and then $\bar{\gamma}_2$, is a closed piecewise C^1 curve, and so

$$\begin{aligned} 0 &= \int_0^1 \langle Z(\sigma(t)), \dot{\sigma}(t) \rangle \, dt \\ &= \int_0^1 \langle Z(\gamma_1(t)), \dot{\gamma}_1(t) \rangle \, dt + \int_0^1 \langle Z(\bar{\gamma}_2(t)) \dot{\bar{\gamma}}_2(t) \rangle \, dt \\ &= \int_0^1 \langle Z(\gamma_1(t)), \dot{\gamma}_1(t) \rangle \, dt - \int_0^1 \langle Z(\gamma_2(t)) \dot{\gamma}_2(t) \rangle \, dt. \end{aligned}$$

Thus the map

$$f(x) = \int_0^1 \langle Z(\gamma(t)), \dot{\gamma}(t) \rangle \, dt$$

where γ is a piecewise C^1 curve in Ω connecting x_0 to x is well defined.

Then let us try to calculate $\partial_{x_i} f(x)$. To do this let us denote by γ a chosen curve from x_0 to x. Let e_i denote the i^{th} standard basis vector

$$\begin{aligned} f(x + h\, e_i) &= \int_0^{1-h} Z((1-h)\gamma(t/(1-h)), \dot{\gamma}(t/(1-h))\rangle \, dt + \int_0^h Z(x + t\, e_i), e_i \rangle \, dt \\ &= \int_0^1 \langle Z(\gamma), \dot{\gamma} \rangle \, dt + \int_0^h Z_i(x + te_i) \, dt \\ &= f(x) + \int_0^h Z_i(x + te_i) \, dt. \end{aligned}$$

So if we calculate the derivative

$$\begin{aligned} \partial_{x^i} f &= \lim_{h \to 0} \frac{1}{h} \left[f(x + h\, e_i) - f(x) \right] \\ &= \lim_{h \to 0} \frac{1}{h} \int_0^h Z_i(x + te_i) \, dt. \\ &= Z_i(x). \end{aligned}$$

Consequently $\nabla f = Z$. $\square$

2.

solution. One of the most useful intuitions to develop about differential forms is that they are sorts of spatial distributions of directed measures. In this case we are integrating along circles, and we want the restricted measures to always integrate to a constant. The length of the circles is linear in $r = \sqrt{x^2 + y^2}$, so we want the magnitude of ω to behave as the inverse of r.

Another thing we want is for the form to restrict to a nonzero form on the circles, so we intuitively want it to follow the circles' directions. The unit "normal" direction is given by the radial form $dr = x\,dx/r + y\,dy/r$. A perpendicular direction to this would be $-y\,dx/r + x\,dy/r$. Then to get the appropriate scaling, we divide once more by r, so we arrive at

$$\omega := -y\,dx/r^2 + x\,dy/r^2.$$

Let us now try to integrate

$$\begin{aligned}
\int_0^1 -\gamma_2(t)\dot{\gamma}_1/r^2 + \gamma_1(t)\dot{\gamma}_2/r^2\,dt &= \int_0^1 -r\sin(2\pi t)\cdot(-r2\pi sin(t))/r^2 + r\cos(2\pi t)\,r2\pi\cos(2\pi t)/r^2\,dt \\
&= \int_0^1 2\pi(\sin^2(2\pi t) + \cos^2(2\pi t))\,dt \\
&= 2\pi.
\end{aligned}$$

Thus ω satisfies the desired property. $\square$

3.

solution.

1. The difficulty in this is to show that addition is well defined. The operation $(\alpha, \beta) \mapsto \alpha + \beta$ is symmetric, and hence abelian, so we need only show that for $\alpha \sim \hat{\alpha}$, the following hold

   $$a\,\alpha \sim a\,\hat{\alpha} \quad \alpha + \beta \sim \hat{\alpha} + \beta.$$

   If $\alpha \sim \hat{\alpha}$ then $\alpha'(0) = \hat{\alpha}'(0)$ but then $a\,\alpha'(0) = a\,\hat{\alpha}'(0)$, so $a\,\alpha \sim a\,\hat{\alpha}$. And similarly $(\alpha+\beta)'(0) = \alpha'(0) + \beta'(0) = \hat{\alpha}'(0) + \beta'(0) = (\hat{\alpha} + \beta)'(0)$.

   Now we must show that the vector space properties hold. Firstly we have an additive identity and additive inverse given by $[x_0]$, and $[-\alpha + 2x_0]$, where x_0 denotes the constant map. As for distributivity, we have that $a([\alpha]+[\beta]) = a[\alpha+\beta] = [a\alpha + a\beta] = a[\alpha] + a[\beta]$.

2. Here we must only show that for $\alpha \sim \gamma$, $\phi \circ \alpha \sim \phi \circ \gamma$. But by the chain rule $(\phi \circ \alpha)'(0) = D\phi_{\alpha(0)} \cdot \alpha'(0) = D\phi_{x_0} \cdot \gamma'(0) = (\phi \circ \gamma)'(0)$.

3. We must once again construct explictly the map. But let us use the map $\Phi_{x_0} : (x_0, e_i) \mapsto [x_0 + te_i]$ defined on the standard basis of $T_{x_0}\mathbb{R}^n$. We will then construct

an explicit inverse, given by $[\gamma] \mapsto (x_0, \gamma'(0))$. Then we must show that these are left and right inverses, i.e. firstly that $(x_0 + te_i)' = e_i$ which is trivial, and then that $\gamma \sim x_0 + t\gamma'(0)$. But this to is trivial as $(x_0 + t\gamma'(0))' = \gamma'(0)$.

As for the translation invariance we first note that $D\tau = I$, so $\Phi_{x_1}(D\tau(v)) = [x_1 + tv]$, while $\Phi_{x_0}(v) = [x_0 + tv]$, so $\tau_*(\Phi_{x_0}(v)) = [\tau(x_0 + tv)] = [x_0 + tv + (x_1 - x_0)] = [x_1 + tv]$. Hence the two are equal.

4. By definition

$$(df)_{x_0}(v) = \lim_{t \to 0} \frac{1}{t}(f(x_0 + tv) - f(x_0)).$$

But this is the same as $(f \circ \gamma)'(0)$ where $\gamma(t) = x_0 + tv$, and $[\gamma] = \Phi(v)$.

5. This should read

$$\tilde{T}_{x_0}V := \{[\gamma] \in \tilde{T}_{x_0}\mathbb{R}^n : \exists \alpha \sim \gamma \text{ and } \alpha(-1,1) \subset V\}.$$

This makes $\tilde{T}_{x_0}V$ well defined. Then clearly $\Phi(x_0, v) = [x_0 + tv]$ takes (x_0, v) to a representative with a member whose image is contained in V, namely the representative $x_0 + tv$. Now given a curve γ contained in V, implies that $x_0 + t\gamma'(0)$ is contained in V, but $\gamma \sim x_0 + t\gamma'(0)$. Hence $\Phi(\gamma'(0)) = [\gamma]$.

$\square$

4.

solution. This is a simple proof by induction. Any linear map A can be decomposed into a product of simple matrices, $A = E_1 \times \cdots \times E_k$. Let us formalise this: Assume for all products of k simple matrices we have that $|E_k \cdots E_1(A)| = |\det(E_k \cdots E_1)||A|$, then we will show that for any product of $k+1$ simple matrices we have that $|E_{k+1}E_k \cdots E_1(A)| = |\det(E_{k+1}E_k \cdots E_1)||A|$, and of course how we will show this is for any simple matrix E, we will show that $|E(A)| = |\det E||A|$.

Simple matrices take one of three forms. A row swap, $S_{i,j}$, with determinant $\det S = -1$. A row scaling $\Lambda_{i,\lambda}$ with $\det \Lambda_{i,\lambda} = \lambda$ and a shear where a multiple of one row is added to another, H with $\det H = 1$. To complete the proof we must show that the property holds only for an n-interval, $I = I_1 \times \cdots \times I_n$, with $I_i = (a_i, b_i)$. By the definition of Lebesgue measure we can cover cover an arbitrary set A, with a disjoint collection of open n-intervals such that their union, U covers A and the $U \setminus A < \epsilon$ for arbitrary ϵ Simultaneous the difference can be covered by a family of disjoint squares who's total volume is 2ϵ. Now A will map a square into a square of uniformly scaled by α , hence the square covering will be mapped into a square covering of measure $\alpha 2\epsilon$. These can be chosen arbitrarily small, so it is sufficient to examine how simple matrices act on an n-interval.

Now $S_{i,j}(I) = I_1 \times \cdots \times I_j \times \cdots \times I_i \times \cdot \times I_n$, and so $|S_{i,j}(I)| = II| = |\det S_{i,j}||I|$. And $\Lambda_{i,\lambda}(I) = I_1 \times \cdots \times \lambda I_i \times \cdots \times I_n$, so $|\Lambda_{i,\lambda}(I)| = |\lambda||I| = |\det \Lambda_{i,\lambda}||I|$. The hardest is a shear, because it doesn't map an interval to an interval. The shear $H_{i,j,\lambda}$ maps the set $I = \prod I_l$ to

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_l \in I_l \text{ for } l \neq j, \lambda x_i + a_j < x_j < \lambda x_i + b_j.$$

In this case then we can apply Fubini to calculate that

$$|H_{i,j,\lambda} \int_{I_1} \cdots \int_{I_n} \int_{\lambda x_i + a_j}^{\lambda x_i + b_j} dx^j \, dx_n \, \dots d\hat{x}^j \, \dots dx^1 = |b_j - a_j| \prod_{l \neq j} |b_l - a_l| = |I| |\det H_{i,j,\lambda}|.$$

□

5.

solution. 1. A linear map is specified by its action on a basis. Let ε_i be the map that takes e_i to 1 and ϵ_j to 0 if $j \neq i$. Now given a dual element α, then $\alpha = \sum_i \alpha(e_i)\epsilon(i)$ to see this consider the action

$$[\alpha - \sum_i \alpha(e_i)\epsilon_i](e_j) = \alpha(e_j) + \sum_i \alpha(e_i)\delta_{ij} = \alpha(e_j) - \alpha(e_j) = 0$$

. To see the linear independence. Suppose that $\sum_i \lambda_i \varepsilon_i(e_j)$ but then $\lambda_j = 0$ for all j.

2. Let f_{ij} be the linear map which takes e_i to f_j and maps e_j to 0 for $j \neq i$. Then let A be a linear map. Let $A(e_i) = \sum_j a_{ij} e'_j$, and consider

$$(A - \sum_{i,j} a_{ij} f_{ij})(e_k) = A(e_k) - \sum_{i,j} a_{ij} f_{ij}(e_k) = \sum_j a_{kj} e'_j - \sum_j a_{kj} e'_j = 0$$

Then for linear independence suppose $\sum_{i,j} a_{ij} f_{ij} = 0$, then

$$\sum_{i,j} a_{ij} f_{ij}(e_k) = \sum_j a_{kj} e'_j = 0.$$

By the basis property of e'_j we have that $a_{kj} = 0$ for all j, but k was also arbitrary.

□

6.

solution. The idea of a homotopy is to construct a family of paths between the values of one map and that of another. In this case we have to construct a continuous family from $X = \mathbb{R}^3 \setminus [0, \infty] \times \{(0, 0)\}$ to a single point. An appropriate point is $(-1, 0, 0)$. Then every ray from this point to every point is contained in X. Hence we can take the homotopy

$$H(t, x) = t(-1, 0, 0) + (1 - t)x,$$

which is the identity for $t = 0$ and is $(-1, 0, 0)$ for $t = 1$. Lastly $H(t, x) \in X$ for all t and x. This can be seen if one of x_2 or x_3 is not equal to zero, then without loss of generality $H(t, x)_2 = 0$ until $t = 1$. If $x_2 = x_3 = 0$ then $x_1 < 0$ but then $H(t, x)_1 < 0$ for all t so is in X. $\square$