Introduction to Differential forms
Spring 2011
Exercise 12 (for Wednesday May. 4)

$\star$**1.** Let (U, φ) be a chart on a smooth manifold M; $\varphi\colon U \to \mathbb{R}^n$, $\varphi = (\varphi_1, \ldots, \varphi_n)$. Show that $(d\varphi_1)_p, \ldots, (d\varphi_n)_p$ is a basis of T_p^*M for $p \in U$. Show also that $\{(d\varphi_I)_p \colon I = (i_1, \ldots, i_k), 1 \le i_1 < \cdots i_k \le n\}$ is a basis of $\mathrm{Alt}^k(T_pM)$, where $d\varphi_I = d\varphi_{i_1} \wedge \cdots \wedge d\varphi_{i_k}$. Moreover, verify that $\varphi^*(dx_k) = d\varphi_k$ for every k.

$\star$**2.** Let $\hat{S}_k \colon \Omega^{k+1}(B^n \times \mathbb{R}) \to \Omega^k(B^n)$ be the operator in the proof of Poincaré homotopy operator, that is,
$\hat{S}_k(\sum_I f_I dt \wedge dx_I + \sum_J g_J dx_J)_x = \sum_I \left(\int_0^1 f_I(x,t)\, d\mathcal{L}^1(t) \right) dx_I$.

(i) Show that $\hat{S}_k(\omega)_x(v_1, \ldots, v_k) = \int_0^1 \omega_{(x,t)}(\frac{\partial}{\partial t}, v_1, \ldots, v_k)\, d\mathcal{L}^1(t)$ for every $\omega \in \Omega^k(B^n \times \mathbb{R})$.

(ii) Let M be a smooth manifold and (U, φ) be a chart on M. Show that $\varphi^* \circ \hat{S}_k \circ ((\varphi \times \mathrm{id}_{\mathbb{R}})^{-1})^* (\omega)_p(v_1, \ldots, v_k) = \int_0^1 \omega_{(x,t)}(\frac{\partial}{\partial t}, v_1, \ldots, v_k)\, d\mathcal{L}^1(t)$ for $\omega \in \Omega^k(U \times \mathbb{R})$, $p \in U$ and $v_1, \ldots, v_k \in T_pM$.

(iii) Show that, for every $k \ge 0$, there exists $\hat{S}_k^M \colon \Omega^{k+1}(M \times \mathbb{R}) \to \Omega^k(M)$ so that $\iota_1^* - \iota_0^* = d\hat{S}_k^M + \hat{S}_k^M d$, where $\iota_j \colon M \to M \times \mathbb{R}$ is the inclusion $p \mapsto (p, j)$.

(iv) Let f and g be smoothly homotopic smooth maps $M \to N$. Show that, for every $k \ge 0$, there exists $S_k \colon \Omega^{k+1}(N) \to \Omega^k(M)$ so that $f^* - g^* = dS_k + S_{k+1} d$.

3. Let M and N be smooth m- and n-manifolds, respectively. Suppose that $f\colon M \to N$ is a continuous map.

(i) Show that there exists atlases $\{(U_i, \varphi_i) \colon i \ge 0\}$ of M and $\{(V_j, \psi_j) \colon j \in J\}$ of N and sets $K_i \subset W_i \subset \bar{W}_i \subset U_i$ for $i \ge 0$ so that for every $i \ge 0$ there exists $j \in J$ so that $fU_i \subset V_j$, and that K_i is compact, W_i open, $\varphi_i U_i = B^m$, and $\psi_j V_j = B^n$ for every $i \ge 0$ and $j \in J$, and that cover $\{U_i\}_{i \ge 0}$ is locally finite, and $\bigcup_i K_i = M$.

(ii) Show that there exists a sequence of maps $g_k \colon M \to N$ so that $g_k | M \setminus W_k = g_{k-1} | M \setminus W_k$, g_k is homotopic to g_{k-1} rel $M \setminus W_k$,[1] and g_k is smooth in a neighborhood of $\bigcup_{i=0}^k K_i$; we take $g_{-1} = f$.

[1] $F\colon X \times [0,1] \to Y$ is a homotopy rel $A \subset X$ if $F(x,t) = x$ for $x \in A$.

(iii) Show that there exists a smooth map $g\colon M \to N$ homotopic to f.

4. An atlas $= \{(U_i, \varphi_i)\colon i \in I\}$ on a smooth manifold M is *positive* if $\det D(\varphi_i \circ \varphi_j^{-1})(x) > 0$ whenever $x \in \varphi_j(U_i \cap U_j)$. Show that the following conditions are equivalent.

(i) A smooth n-manifold M has an n-form $\omega \in \Omega^k(M)$ so that $\omega_p \neq 0$ for every $p \in M$.

(ii) A smooth manifold M has a positive atlas.

$\star$**5.**

(i) Calculate $H^k(S^n)$ for all $n \geq 0$ and $k \geq 0$; $S^0 = \{\pm 1\}$.

(ii) Calculate $H^k(S^2 \times S^1)$ for all $k \geq 0$.

6. Let $\alpha\colon [0,1] \to \bar{B}^2$ and $\beta\colon [0,1] \to \bar{B}^2$ be injective paths so that $\alpha(0) = -e_1$, $\alpha(1) = e_1$, $\beta(0) = -e_2$, and $\beta(1) = e_2$. Show that $\alpha[0,1] \cap \beta[0,1] \neq \emptyset$.